Projective tensor products, injective tensor products, and dual relations on operator spaces.

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PROJECTIVE TENSOR PRODUCTS,
INJECTIVE TENSOR PRODUCTS,
AND DUAL RELATIONS ON OPERATOR SPACES

by

Fuhua Chen

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Abstract

For operator space $V$ and $W$, there are different ways to define an operator space structure on the algebraic tensor product $V \otimes W$, among which the projective tensor product, denoted by $V \widehat{\otimes} W$, and the injective tensor product, denoted by $V \bar{\otimes} W$, take an critical role for their “Projectivity” and “Injectivity”. In this thesis, we discussed some fundamental properties for these two tensor products.

Among those properties of projective tensor products, the Fubini theorem is especially interesting and useful. Let $H$ and $K$ be Hilbert spaces. Then we have a complete isometry $B(H) \widehat{\otimes} B(K) \cong B(H \otimes_2 K)$. This property can be extended to von Neumann algebras $\mathcal{R} \subseteq B(H)$ and $\mathcal{S} \subseteq B(K)$, i.e., $\mathcal{R} \widehat{\otimes} \mathcal{S} \cong (\mathcal{R} \otimes \mathcal{S})^*$ is a complete isometry, which is due to E. G. Effros and Z.-J. Ruan. We present a detailed proof for this result and its extension in Chapter 5.

Let $X, Y$ and $Z$ be linear spaces over $\mathbb{F}$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). It is well known that for every bilinear mapping $B \in BL(X \times Y, Z)$, there exists a unique linear mapping $\tilde{B} \in L(X \otimes Y, Z)$ such that $B(x, y) = \tilde{B}(x \otimes y)$ for all $x \in X$ and $y \in Y$. The correspondence $B \mapsto \tilde{B}$ is an isomorphism between the vector spaces $BL(X \times Y, Z)$ and $L(X \otimes Y, Z)$. Then an interesting question is that, for operator spaces $V, W$ and $X$, which subspace of $BL(V \times W, X)$ can be identified with $CB(V \widehat{\otimes} W, X)$, and which subspace of $BL(V \times W, X)$ can be identified with $CB(V \bar{\otimes} W, X)$. About the first part of the question, it has been known that $CB(V \widehat{\otimes} W, X) \cong JCB(V \times W, X)$. The answer to the second part of the question is not clear. Z.-J. Ruan proved that, for the particular case $Z = \mathbb{C}$, $(V \bar{\otimes} W)^* \cong I(V, W^*)$ is a complete isometry if and only if $W$ is completely locally reflexive. We present a detailed proof for this property in
Chapter 6. Some examples of operator space with completely local reflexivity are also provided.
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CHAPTER 1

Introduction

For operator spaces $V$ and $W$, there are different ways to define operator space structure on the algebraic tensor product $V \otimes W$, in which the projective tensor product $V \hat{\otimes} W$ and injective tensor product $V \hat{\otimes} W$ are especially important for their "projectivity" and "injectivity", respectively. In this thesis, many comprehensive properties for these two tensor products are discussed, most of which are proved.

For any Hilbert spaces $H$ and $K$, there is a natural complete isometry

$$B(H) \hat{\otimes} B(K) \cong B(H \otimes_2 K)$$

This property can be extended to von Neumann algebras $\mathcal{R} \subseteq B(H)$ and $\mathcal{S} \subseteq B(K)$. This extension is due to E. G. Effros and Z.-J. Ruan. In this thesis, detailed proofs for these two results are presented.

If $V$ is a complete operator space, then $V^*$ has a dual realization on a Hilbert space $H$, i.e., there is an embedding $\varphi : V \rightarrow B(H)$ that is a weak* homeomorphic completely isometric injection. Let $V^*$ and $W^*$ be the dual operator spaces of complete operator spaces $V$ and $W$, respectively, with dual realizations $\pi_1 : V^* \rightarrow B(H)$ and $\pi_2 : W^* \rightarrow B(K)$. We have two dual tensor products associated with these representations. On one hand, the normal spatial tensor product $V^* \hat{\otimes} W^*$ is defined to be the weak* closure of $V^* \otimes W^*$ in $B(H \otimes_2 K)$, where $B(H \otimes_2 K) = (B(H) \hat{\otimes} B(K))^*$. On the other hand, the normal Fubini tensor product $V^* \overline{\otimes}_F W^*$ is defined by

$$V^* \overline{\otimes}_F W^* = \{ u \in B(H \otimes_2 K) : (\omega_1 \otimes id)(u) \in W^* \text{ and } (id \otimes \omega_2)(u) \in V^* \text{ for all } \omega_1 \in B(H)^*, \omega_2 \in B(K)^* \}.$$

The relation between the normal spatial tensor product and the normal Fubini tensor product is revealed in [13]. We give all the proofs in detail in Chapter 5.

Let $X, Y$ and $Z$ be linear spaces over $F$ ($F = \mathbb{C}$ or $\mathbb{R}$). It is well known that for every bilinear mapping $B \in BL(X \times Y, Z)$, there exists a unique linear mapping $\tilde{B} \in L(X \otimes Y, Z)$ such that $B(x, y) = \tilde{B}(x \otimes y)$ for all $x \in X$ and $y \in Y$. The correspondence $B \rightarrow \tilde{B}$ is an isomorphism between the vector spaces $BL(X \times Y, Z)$. 

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and $L(X \otimes Y, Z)$. Then an interesting question is that which subspace of $BL(V \times W, X)$ can be identified with $CB(V \hat{\otimes} W, X)$, and which subspace of $BL(V \times W, X)$ can be identified with $CB(V \hat{\otimes} W, X)$.

About the first part of the question, it was proved by D. Blecher and V. Paulsen in 1991 (see [3]) that $CB(V \hat{\otimes} W, X) \cong JCB(V \times W, X)$, where $JCB(V \times W, X)$ is the linear space of jointly completely bounded bilinear mappings. The answer to the second part of the question is not clear. Even in the Banach space case, we only know the answer for the particular case $Z = C$, i.e., $(X \hat{\otimes} Y)^* \cong B_I(X \times Y) \cong I(X, Y^*)$ for Banach spaces $X$ and $Y$. However, this motivates people to look for a similar result for operator space case. In paper [12], Z.-J. Ruan proved that $(V \hat{\otimes} W)^* \cong I(V, W^*)$ if and only if $W$ is completely locally reflexive. We present a detailed proof of this property in Chapter 6. Some examples of operator space with completely local reflexivity are shown.

Having a review of the Banach space $B(H)$ of all bounded linear operators on a Hilbert space $H$ is very useful for understanding abstract operator spaces. In Chapter 3, after reviewing representations for the spaces $K(H)$ of all compact operators and $TC(H)$ of all trace class operators, which are cited from [25], we prove that for any Hilbert space $H$, $B(H)$ is isometric to $\mathcal{M}_I$ for some index set $I$, where $\mathcal{M}_I$ is the normed matrix space with a norm defined by

$$\|F\| = \sup\{\|F'\| : F' \text{ is a finite submatrix of } F\}$$

(see Theorem 3.4.5). The double dual relation $K(H)^{**} \cong B(H)$ (see Theorem 3.4.6) is well-known. We prove two aspects of the relation. First, the correspondence $B(H) \longrightarrow K(H)^{**}$, $x \longmapsto f_x$, is multiplicative, where the multiplication in $B(H)$ is operator composition and the multiplication in $K(H)^{**}$ is the Arens product (see Theorem 3.4.7). Second, the correspondence between $B(H)$ and $K(H)^{**}$ preserves the corresponding module structures (see Theorem 3.4.8 and Theorem 3.4.9).

Chapter 4 is a brief introduction to concrete operator space and abstract operator space. We will see that every concrete operator space is an abstract operator space (Proposition 4.1.7), and the converse is also true in the sense that every abstract operator space is completely isometric to a subspace of $B(H)$ for some Hilbert space $H$. This representation theorem was proved by Z.-J. Ruan in 1988 (see [18]).
CHAPTER 2

Preliminaries

In this chapter, we will collect some notation and conventions which will be used throughout this thesis.

2.1. Banach spaces

Given a vector space $V$ over a subfield $F$ of the complex numbers $\mathbb{C}$ such as the complex numbers themselves or the real or rational numbers, a semi-norm on $V$ is a function $p : V \rightarrow [0, \infty)$, $x \mapsto p(x)$, with the following properties:

1. $p(au) = |a|p(u)$ (positive homogeneity or positive scalability);
2. $p(u + v) \leq p(u) + p(v)$ (triangle inequality or subadditivity).

A norm is a semi-norm with the additional property

3. $p(v) = 0$ if and only if $v$ is the zero vector (positive definiteness).

A normed (vector) space is a pair $(V, \| \cdot \|)$, where $V$ is a vector space and $\| \cdot \|$ is a norm on $V$. A Banach space is a normed space that is complete with respect to the metric defined by the norm.

**Theorem 2.1.1.** If $(X, \| \cdot \|)$ is a normed space and $Y$ is a vector subspace of $X$, then the norm closure $\overline{Y}$ is still a subspace of $X$. Moreover, if $X$ is a Banach space, then $\overline{Y}$ is a Banach space.

Let $X$ be a Banach space, $M$ a closed subspace of $X$, and $X/M$ the quotient vector space. Let $Q : X \rightarrow X/M$, $x \mapsto x + M$, be the natural quotient mapping. Define a function $\| \cdot \| : X/M \rightarrow [0, \infty)$ by $\|x + M\| = \inf\{\|x + y\| : y \in M\}$ for all $x \in X$. Then $\| \cdot \|$ is a norm on $X/M$. Moreover, $X/M$ is a Banach space.

A linear functional on a normed space $X$ is a linear map $f : X \rightarrow \mathbb{C}$. $f$ is said to be bounded if $\|f\| = \sup_{\|f\| \leq 1} |f(x)| < \infty$. All bounded linear functionals on $X$ form a linear space and $\| \cdot \|$ defines a norm on this space. Denote by $X^*$ the normed space of all those bounded linear functionals equipped with the above norm. Then $X^*$ is a
2.3. C*-ALGEBRAS

Banach space, called the dual space of $X$. For $X^*$, we can also define its dual $(X^*)^*$, usually denoted by $X^{**}$. $X$ is said to be reflexive if $X = X^{**}$ as normed spaces.

**Theorem 2.1.2. (Hahn-Banach Theorem)** Let $X$ be a vector space over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), $p : X \rightarrow [0, \infty)$ a seminorm on $X$, $M$ a subspace of $X$, and $f$ a linear functional on $M$ satisfying $f(y) \leq p(y)$ for all $y \in M$. Then there exists a linear functional $F$ on $X$ such that the restriction of $F$ on $M$ is equal to $f$, and $|F(x)| \leq p(x)$ for all $x \in X$.

**Corollary 2.1.3.** If $V$ is a normed vector space with subspace $U$ (not necessarily closed) and if $f : U \rightarrow \mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is continuous and linear, then there exists an extension $g : V \rightarrow \mathbb{F}$ of $f$ which is also continuous and linear, and which has the same norm as $f$.

2.2. Hilbert spaces

If $X$ is a vector space over $\mathbb{F}$, a semi-inner product on $X$ is a function $u : X \times X \rightarrow \mathbb{F}$ such that for all $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in X$, the following are satisfied:

1. $u(\alpha x + \beta y, z) = \alpha u(x, z) + \beta u(y, z)$,
2. $u(x, y) = \overline{u(y, x)}$,
3. $u(x, x) \geq 0$.

An inner product on $X$ is a semi-inner product that also satisfies the following:

4. If $u(x, x) = 0$, then $x = 0$.

In this thesis, every inner product is denoted by $\langle \cdot, \cdot \rangle$. Every inner product on a real or complex vector space $H$ gives rise to a norm $\| \cdot \|$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in H$. $H$ is called a Hilbert space if it is complete with respect to the norm induced by the inner product.

2.3. C*-algebras

Let $K$ be a field, and let $A$ be a vector space over $K$. If there is a binary operation on $A$, called multiplication (denoted by $xy$ for any $x, y \in A$), satisfying

$$(x + y)z = xz + yz \text{ and } x(y + z) = xy + xz$$

for any three elements $x, y, z$ of $A$, and all elements ("scalars") $a$ and $b$ of $K$, then $A$ is said to be an algebra over $K$, and $K$ is the base field of $A$. Multiplication is not necessarily associative.
2.4. ARENS PRODUCTS

Instead of a field, $K$ can be a commutative ring with bilinear multiplication again satisfying the above identities. In this case, $A$ is a $K$-algebra, and $K$ is the base ring of $A$.

Let $A$ and $B$ be algebras. A linear mapping $f : A \to B$ is said to be a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in A$. If, further, $f$ is both injective and surjective, then $f$ is said to be an isomorphism. In this case, $A$ and $B$ are said to be isomorphic. A Banach algebra is a complex normed algebra $A$ which is complete (as a normed space) and satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

A $*$-algebra is a complex algebra $A$ with a conjugate linear involution $*$ (called the adjoint) which is an anti-isomorphism. That is, for all $a, b \in A$ and $\alpha \in \mathbb{C}$,

1. $(a + b)^* = a^* + b^*$;
2. $(\alpha a)^* = \overline{\alpha} a^*$;
3. $a^{**} = a$;
4. $(ab)^* = b^*a^*$.

Let $A$ and $B$ be $*$-algebras. A homomorphism $f : A \to B$ is said to be a $*$-homomorphism if $f(x^*) = f(x)^*$ for all $x \in A$. A $*$-homomorphism is said to be a $*$-isomorphism if it is an isomorphism.

A $C^*$-algebra is a Banach $*$-algebra with the additional norm condition $\|a^*a\| = \|a\|^2$ for all $a \in A$. A Banach algebra is said to be unital if it admits a multiplicative unit $e$ and $\|e\| = 1$.

2.4. Arens Products

Let $X, Y,$ and $Z$ be normed spaces over $\mathbb{C}$ and $m$ a bounded bilinear map from $X \times Y$ into $Z$. The two adjoint maps of $m$, namely $m^* : Z^* \times X \to Y^*$ and $m_* : Y \times Z^* \to X^*$ are defined as follows.

For $f \in Z^*, x \in X,$ and $y \in Y$, let

$$m^*(f, x)(y) = f(m(x, y)).$$

$$m_*(y, f)(x) = f(m(x, y)).$$

In particular, if $X = Y = Z$ and $m : X \times X \to X$, then we have

$$m^* : X^* \times X \to X^*,$$
The counterparts of $m_*$, $m_*$, and $m_{**}$ can be similarly defined. It is easy to see that both $m_{**}$ and $m_{***}$ are natural extensions of $m$.

Let $A$ be a Banach algebra and let $m : A \times A \to A$ be the multiplication on $A$. Then the first Arens product on $A^{**}$ is $m_{***}$, denoted by $*_1$. The second Arens product on $A^{**}$ is $m_{***}$, denoted by $*_2$. $A$ is called *Arens regular* if the two Arens products agree on $A^{**}$. Due essentially to Sherman and Takeda's work, any $C^*$-algebra is Arens regular (see [6, page 310]). This is because, if we identify a $C^*$-algebra $A$ with its image under universal representation, then $A^{**}$ may be identified with the closure of $A$ under weak operator topology (see [23] and [24]). Since the two Arens products are equal when restricted in $A$, they must also be equal in $A^{**}$, the weak operator closure of $A$.

For Arens regularity, we also have the following theorem (see [26]).

**Theorem 2.4.1.** *Let $X$ be an arbitrary Banach space and $K(X)$ the operator algebra consisting of compact linear operators $u : X \to X$. Then the algebra $K(X)$ is Arens regular if and only if the space $X$ is reflexive.*

### 2.5. Space of Measures

Let $S$ denote an arbitrary nonvoid set. A nonvoid family $\mathcal{P}$ of subsets of $S$ is called a *ring* of sets if $A \cup B \in \mathcal{P}$ and $A \cap B' \in \mathcal{P}$ whenever $A, B \in \mathcal{P}$, where $B'$ denotes the complement of $B$. If the set $S$ itself is in the ring $\mathcal{P}$, then $\mathcal{P}$ is called an *algebra* of sets. A *$\sigma$-algebra* $\mathcal{P}$ of sets is an algebra such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{P}$ whenever $\{A_1, A_2, \ldots, A_k, \ldots\} \subseteq \mathcal{P}$.

We denote by $\emptyset$ the empty set. A *finitely additive measure* $\mu$ is a nonnegative extended real-valued set function defined on a ring $\mathcal{P}$ of subsets of a nonvoid set $S$ such that:

1. $\mu(\emptyset) = 0$;
2. $\mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)$ for all pairwisely disjoint $A_k \in \mathcal{P}$.

If property (2) can be replaced by the stronger property

3. $\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ for all pairwisely disjoint $A_k \in \mathcal{P}$,
then \( \mu \) is called a \textit{countably additive measure} or simply a \textit{measure}. If \( \mu(A) < \infty \) for all \( A \in \mathcal{P} \), then \( \mu \) is said to be finite. A complex-valued set function \( \mu \) satisfying (1) and (3) is called a \textit{complex measure}.

Let \( X \) be any set and let \( \Omega \) be a \( \sigma \)-algebra of subsets of \( X \). Then \( (X, \Omega) \) is called a \textit{measurable space}. If \( \mu \) is a positive measure on \( (X, \Omega) \), then \( (X, \Omega, \mu) \) is called a \textit{measure space}.

Let \( X \) be a topological space, and \( O \) be the family of all open subsets of \( X \). Then the family of Borel sets in \( X \) is defined as the smallest \( \sigma \)-algebra of sets in \( X \) containing \( O \), called the \textit{Borel} \( \sigma \)-\textit{algebra} on \( X \). If \( B \) is the Borel \( \sigma \)-algebra on some topological space \( X \), then a measure \( \mu : B \rightarrow [0, \infty] \) is said to be a Borel measure.

Let \( B \) be the Borel \( \sigma \)-algebra on a topological space \( X \) and \( \mu \) a Borel measure defined on \( B \). Suppose that:

1. for every open set \( V \), we have
   \[
   \mu(V) = \sup\{\mu(F) : F \text{ is compact and } F \subseteq V\};
   \]
2. for all \( A \in \mathcal{B} \), we have
   \[
   \mu(A) = \inf\{\mu(V) : V \text{ open and } A \subseteq V\}.
   \]

Then \( \mu \) is said to be \textit{regular}.

Let \( X \) be a locally compact Hausdorff space and \( \mathcal{B} = \mathcal{B}(X) \), the Borel \( \sigma \)-algebra on \( X \). Let \( \mu : \mathcal{B} \rightarrow \mathbb{C} \) be a complex measure. For \( E \in \mathcal{B} \), we define
\[
|\mu|(E) = \sup\{\sum_{i=1}^{n} |\mu(E_i)| : E = \bigcup_{i=1}^{n} E_i, E_i \in \mathcal{B} \text{ are pairwisely disjoint}\}.
\]
It is known that for such \( \mu, |\mu| \) has only finite values. Then \( |\mu(E)| \leq |\mu|(E) \) for all \( E \in \mathcal{B} \) and \( |\mu| : \mathcal{B} \rightarrow [0, \infty] \) is a positive measure. \( \mu \) is said to be regular if \( |\mu| \) is regular.

We use \( M(X) \) to denote all complex regular Borel measures \( \mu \) on \( X \). Define
\[
\|\mu\| = |\mu|(X).
\]
Then \( (M(X), \|\cdot\|) \) is a Banach space. We use \( C_0(X) \) to denote all continuous functions \( f \) defined on \( X \) such that for any \( \epsilon > 0 \), there is a compact subset \( K \subseteq X \) with \( |f(x)| < \epsilon \) for all \( x \) outside of \( K \). Then \( C_0(X) \) is a normed space with the usual
supremum norm. For $\mu \in M(X)$, define $\tau(\mu) : C_0(X) \to \mathbb{C}$ by

$$\tau(\mu)(f) = \int_X f(x) d\mu$$

for all $f \in C_0(X)$. Then $\tau(\mu) \in C_0(X)^*$. Moreover, we have the following theorem.

**Theorem 2.5.1. (Riesz Representation Theorem)** The map

$$\tau : M(X) \to C_0(X)^*, \mu \mapsto \tau(\mu),$$

is a linear isometry from $M(X)$ onto $C_0(X)^*$.

### 2.6. Algebraic Tensor Products

Let $X, Y$ and $Z$ be linear spaces. A mapping $B : X \times Y \to Z$ is said to be **bilinear** if it is linear in each variable. In case that $Z = \mathbb{C}$, $B$ is called a **bilinear form**.

We use $BL(X \times Y, Z)$ to denote all the bilinear mappings from $X \times Y$ to $Z$, and use $BL(X \times Y)$ to denote $BL(X \times Y, \mathbb{C})$ in short.

For any normed space $X$, $X^*$ denotes the Banach dual of $X$. Let $E$ and $F$ be two normed spaces and $\varphi : E \to F$ a bounded linear mapping. The dual mapping $\varphi^* : F^* \to E^*$ of $\varphi$ is defined by $\varphi^*(f)(x) = f(\varphi(x))$ for all $f \in F^*$ and $x \in E$.

Then, $\|\varphi^*\| = \|\varphi\|$. We use $B(E, F)$ to denote all bounded linear mappings from $E$ to $F$. Then $B(E, F)$ is a normed space with the operator norm. In case that $F$ is complete, $B(E, F)$ is also complete.

Given a linear space $X$ and $n \in \mathbb{N}$, $M_n(X)$ denotes the linear space of all $n \times n$ matrices over $X$. In case that $X$ is a normed space, if a norm is given in $M_n(X)$, then we denote by $M_n(X)$ the normed matrix space to distinguish it from the corresponding linear space.

For $x \in X$ and $y \in Y$, define the algebraic tensor product of $x$ and $y$, denoted by $x \otimes y$, to be the linear form on $BL(X \times Y)$ defined by $(x \otimes y)(\varphi) = \varphi(x, y)$ for all $\varphi \in BL(X \times Y)$. Then $x \otimes y$ is a linear functional on $BL(X \times Y)$. Let $X \otimes Y$ be the linear span of $\{x \otimes y : x \in X, y \in Y\}$, called the **algebraic tensor product** of $X$ and $Y$. Then $X \otimes Y$ is a subspace of the dual space of $BL(X \times Y)$.

Let $H$ and $K$ be two Hilbert spaces with inner products $\langle , \rangle_H$ and $\langle , \rangle_K$, respectively. Construct the algebraic tensor product of $H$ and $K$, and then turn this vector
space into an inner product space by defining
\[ \langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle_H \langle k_1, k_2 \rangle_K \]
for all \( h_1, h_2 \in H \) and \( k_1, k_2 \in K \). Then extend it linearly. Finally, take the completion under this inner product. The resulted space, denoted by \( H \otimes_2 K \), is the Hilbert tensor product of \( H \) and \( K \).

The primary application of tensor products is to linearize bilinear mappings. To explain this, let \( A \in BL(X \times Y) \). Each tensor \( u \in X \otimes Y \) acts as a linear functional on the space \( BL(X \times Y) \) and so we may define a mapping \( \tilde{A} : X \otimes Y \rightarrow \mathbb{C} \) by \( \tilde{A}(u) = u(A) \) for all \( u \in X \otimes Y \). Then \( \tilde{A} \) is a linear functional on \( X \otimes Y \). On the other hand, it is easy to see that if \( \Psi \) is a linear functional on the tensor product \( X \otimes Y \), then the composition of \( \Psi \) with the bilinear mapping \( (x, y) \mapsto x \otimes y \) is a bilinear form \( A \) on \( X \times Y \) for which \( \tilde{A} = \Psi \). Thus, the bilinear forms on \( X \times Y \) are in one-to-one correspondence with the linear functionals on \( X \otimes Y \). For general case, let \( L(X \otimes Y, Z) \) denote the linear space of all linear mappings from \( X \otimes Y \) to \( Z \). Then we have the following theorem.

**Theorem 2.6.1.** For every bilinear mapping \( A : X \times Y \rightarrow Z \), there exists a unique linear mapping \( \tilde{A} : X \otimes Y \rightarrow Z \) such that \( A(x, y) = \tilde{A}(x \otimes y) \) for all \( x \in X \) and \( y \in Y \). The correspondence \( A \mapsto \tilde{A} \) is an isomorphism between the vector spaces \( BL(X \times Y, Z) \) and \( L(X \otimes Y, Z) \).
In this chapter, we will mainly discuss the Banach algebra $B(H)$ of all bounded linear operators on a Hilbert space $H$. We will especially introduce two subspaces $K(H)$ of compact operators and $TC(H)$ of trace class operators. Most of the results are cited from Takesaki's book [25].

3.1. Introduction to $B(H)$

In this section, we always denote by $B(H)$ the Banach algebra of all bounded linear operators on a Hilbert space $H$, and denote by $K(H)$ the subspace of $B(H)$ consisting of all compact operators. An operator $T \in B(H)$ is called of finite rank if its range $T(H)$ is finite dimensional. The set of all finite rank operators on $H$ denoted by $F(H)$ is norm dense in $K(H)$. $T \in B(H)$ is called positive if $\langle Th, h \rangle \geq 0$ for all $h \in H$. We denote by $T^*$ the adjoint of $T$. $T$ is called self-adjoint or hermitian if $T^* = T$. It is the case if and only if $\langle Th, h \rangle \in \mathbb{R}$ for all $h \in H$. Thus, every positive operator is self-adjoint. $T$ is called normal if $T^* T = T T^*$. $T$ is called a partial isometry if $\|Th\| = \|h\|$ for all $h \in (\ker T)^\perp$. $T$ is called unitary if $T^* T = I$, where $I$ is the identity map on $H$.

**Theorem 3.1.1.** (Polar Decomposition [5]) If $T \in B(H)$, then there is a partial isometry $U$ on $H$ such that $T = U |T|$, where $|T| = (T^* T)^{1/2}$. Moreover, $\ker T = \ker U$ and $\text{ran} U = \text{cl}(\text{ran} T)$. If $T$ is invertible or normal, then $U$ can be chosen to be unitary.

**Definition 3.1.2.** A map $B : H \times H \to \mathbb{C}$ is called a sesquilinear form if

(a) $B(\alpha \xi, \eta) = \alpha B(\xi, \eta) = B(\xi, \bar{\alpha} \eta)$ ($\xi, \eta \in H$, $\alpha \in \mathbb{C}$);
(b) $B(\xi_1 + \xi_2, \eta) = B(\xi_1, \eta) + B(\xi_2, \eta)$ ($\xi_1, \xi_2, \eta \in H$);
(c) $B(\xi, \eta_1 + \eta_2) = B(\xi, \eta_1) + B(\xi, \eta_2)$ ($\xi, \eta_1, \eta_2 \in H$).
3.1. INTRODUCTION TO $B(H)$

A sesquilinear form $B$ is said to be positive if $B(\xi, \xi) \geq 0$ for all $\xi \in H$; bounded if

$$\|B\| = \sup \{|B(\xi, \eta)| : \|\xi\| \leq 1, \|\eta\| \leq 1\} < \infty.$$  \hfill (1)

We denote by $SB(H)$ the linear space of all bounded sesquilinear forms on $H$ with the natural pointwise linear operations. Then $(SB(H), \|\cdot\|)$ is a Banach space.

**Theorem 3.1.3.** $B(H) \cong SB(H)$ as Banach spaces via the map $T \leftrightarrow B$ between $B(H)$ and $SB(H)$ determined by

$$(T\xi|\eta) = B(\xi, \eta) \quad (\xi, \eta \in H).$$

Furthermore, $T$ is positive if and only if $B$ is positive.

**Definition 3.1.4.** Let $H$ be a Hilbert space and let $I$ be the identity map on $H$. The spectrum of $x \in B(H)$ is defined to be the set

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda I \text{ is not invertible}\}.$$

**Definition 3.1.5.** Let $X$ be a set, $\Omega$ be a $\sigma$-algebra of subsets of $X$, and $H$ be a Hilbert space. A spectral measure for $(X, \Omega, H)$ is a function $E : \Omega \to B(H)$ such that

(a) for each $\Delta \in \Omega$, $E(\Delta)$ is a projection;
(b) $E(\phi) = 0$ ($\phi$ denotes the empty set) and $E(X) = I$, the identity mapping on $X$;
(c) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for $\Delta_1, \Delta_2 \in \Omega$;
(d) if $\{\Delta_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in $\Omega$, then

$$E\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} E(\Delta_n).$$

in the strong operator topology.

By [5, Theorem IX.1.10], if $E$ is a spectral measure for $(X, \Omega, H)$ and $\varphi : X \to \mathbb{C}$ is a bounded $\Omega$-measurable function, then there is a unique operator $A$ in $B(H)$ such that if $\varepsilon > 0$ and $\{\Delta_1, \ldots, \Delta_n\}$ is an $\Omega$-partition of $X$ with $\sup\{|\varphi(x) - \varphi(x')| : x, x' \in$
\[ \Delta_k \] < \varepsilon \text{ for all } 1 \leq k \leq n, \text{ then}
\[ \| A - \sum_{k=1}^{n} \varphi(x_k)E(\Delta_k) \| < \varepsilon \]
for all choices \( \{x_1, x_2, \ldots, x_n\} \) with \( x_k \) in \( \Delta_k \) (\( 1 \leq k \leq n \)). The operator \( A \) obtained in this way is called the integral of \( \varphi \) with respect to \( E \) and is denoted by \( \int \varphi dE \). For normal operators, we have the following proposition (see [5, Theorem IX.2.2]).

**Theorem 3.1.6. (Spectral Decomposition)** If \( N \) is a normal operator on \( H \), then there is a unique spectral measure \( E \) on Borel subsets of \( \sigma(N) \) such that \( N = \int zdE(z) \).

With the concept of spectral measure, we can decide whether a normal operator is compact by the following theorem (see [5, Proposition IX.4.1]).

**Theorem 3.1.7.** If \( N \) is a normal operator and \( N = \int zdE(z) \) is as in Theorem 3.1.6, then \( N \) is compact if and only if for every \( \varepsilon > 0 \), \( E(\{z : |z| > \varepsilon\}) \) has finite rank.

For self-adjoint compact operators, the spectral decomposition has a very simple form as follows (see [5, Theorem II.5.1 and Corollary II.5.3]).

**Theorem 3.1.8.** If \( T \) is a self-adjoint compact operator on a Hilbert space \( H \), then \( T \) has only countably many eigenvalues. If \( \{\lambda_n\}_{n \in \Lambda} \) are the distinct nonzero eigenvalues of \( T \) (maybe finitely many), and \( P_n \) is the orthogonal projection of \( H \) onto \( \ker(T - \lambda_n I) \), then for all \( n \in \Lambda, \lambda_n \in \mathbb{R} \) with \( \dim(\ker(T - \lambda_n I)) < \infty \), \( P_nP_m = P_mP_n = 0 \) if \( n \neq m \), and
\[ T = \sum_{n \in \Lambda} \lambda_n P_n, \]
where the series converges to \( T \) in the norm topology in \( B(H) \).

### 3.2. Compact Operators

In this section, we will mainly introduce the representation theorem for compact operators (see Theorem 3.2.2).

For each pair \( \xi, \eta \) in \( H \), we define the operator \( t_{\xi, \eta} \) of rank one by
\[ t_{\xi, \eta} \zeta = \langle \zeta | \eta \rangle \xi, \quad \zeta \in H. \]
On the other hand, every operator of rank one is of this form. To see this, let $T$ be an operator of rank one. Then there exists a $\xi \in H$ and $f \in H^*$ such that $Th = f(h)\xi$ ($h \in H$). By Riesz representation theorem, there exists an $\eta \in H$ such that $f(h) = \langle h|\eta \rangle$. Then $Th = \langle h|\eta \rangle \xi = t_{\xi,\eta}h$. Thus, $T = t_{\xi,\eta}$. The following properties can be easily proved.

**Proposition 3.2.1.** Let $\xi, \xi_1, \xi_2, \eta, \eta_1, \eta_2 \in H$, $u \in B(H)$ and $\alpha \in \mathbb{C}$. Then

(a) $t_{\xi_1+\xi_2,\eta} = t_{\xi_1,\eta} + t_{\xi_2,\eta}$
(b) $t_{\xi,\eta_1+\eta_2} = t_{\xi,\eta_1} + t_{\xi,\eta_2}$
(c) $t_{\alpha \xi,\eta} = \alpha t_{\xi,\eta}$, $t_{\xi,\alpha \eta} = \overline{\alpha} t_{\xi,\eta}$
(d) $t_{u \xi,\eta} = ut_{\xi,\eta}$, $t_{\xi,u \eta} = t_{\xi,\eta}u^*$

**Theorem 3.2.2.** Let $H$ be an infinite dimensional Hilbert space and $T$ a compact operator on $H$. Then there exist two normalized orthogonal systems $\{\xi_n\}$, $\{\eta_n\}$ in $H$ and a positive sequence $\{\alpha_n\}$ in $c_0$ such that

$$T = \sum_n \alpha_n t_{\xi_n,\eta_n}$$

and $\|T\| = \|\{\alpha_n\}\|_\infty$. \hspace{1cm} (3)

If $T$ is self-adjoint in addition, then $T$ is represented in the form

$$T = \sum_n \alpha_n t_{\xi_n,\eta_n}$$

for some normalized orthogonal system $\{\xi_n\}$ and a real sequence $\{\alpha_n\}$ in $c_0$.

**Proof.** First, we suppose that $T$ is self-adjoint. By Theorem 3.1.8, $T$ has a spectral decomposition

$$T = \sum_n \lambda_n P_n,$$

where $\{\lambda_i\}$ are all non-zero eigenvalues of $T$ which are real numbers for all $i$, $P_i$ is the projection of $H$ onto $\ker(T - \lambda_i I)$ satisfying $P_i P_{i'} = P_{i'} P_i = 0$ if $i \neq i'$. The equality (5) holds in the norm topology. So, $\{\lambda_n\} \in c_0$ and $\|T\| = \sup |\lambda_n|$.

For each $i$, take $S_i = \{\xi_{ik} : k = 1, ..., m_i\}$ ($m_i = \dim(\ker(T - \lambda_i I))$) to be an orthonormal basis of $\ker(T - \lambda_i)$. Then we have

$$P_i = \sum_{k=1}^{m_i} t_{\xi_{ik},\xi_{ik}}.$$
Since $S_i \perp S_i'$ if $i \neq i'$, $\bigcup S_i$ is a normalized orthogonal system in $H$ and $T = \sum_{n}^{\infty} \lambda_n \sum_{k=1}^{m} t_{\xi_k, \xi_n}$. Now, we rearrange the indices and, if necessary, make $\bigcup S_i$ into a orthonormal system $\{\xi_n\}$. Then $T$ has the form (4).

For the general case, let $T = U|T|$ be the polar decomposition of $T$, where $U$ is a partial isometry. Then $|T| = U^*U|T| = U^*T$ is compact. Applying the above arguments to $|T|$, we represent $|T|$ in the form $|T| = \sum_{n}^{\infty} \alpha_n \xi_n, \xi_n$. Note that all $\alpha_n$ are non-zero eigenvalues of $|T|$ or zero and that $|T|$ is positive, so $\alpha_n \geq 0$ for all $n \in \mathbb{N}$. Now, by Proposition 3.2.1(d), we have

$$T = U|T| = \sum_{n}^{\infty} \alpha_n U\xi_n, \xi_n = \sum_{n}^{\infty} \alpha_n U\xi_n, \xi_n.$$

Let $\eta_n = U\xi_n$ $(n \in \mathbb{N})$. Then $\{\eta_n\}$ and $\{\xi_n\}$ are the desired normalized orthogonal systems. To see that $\{\eta_n\} = \{U\xi_n\}$ is a normalized orthogonal system, note that for each $\xi_n$, there exists a $\lambda_i \neq 0$ such that $|T|\xi_n = \lambda_i \xi_n$. Or equivalently, $\xi_n = \frac{1}{\lambda_i}|T|\xi_n$.

Now,

$$\langle U\xi_m|U\xi_n \rangle = \langle \xi_m|U^*U\xi_n \rangle = \langle \xi_m|U^* U \frac{1}{\lambda_i}|T|\xi_n \rangle$$

$$= \langle \xi_m|U^*U|T|\xi_n \rangle = \langle \xi_m|\frac{1}{\lambda_i}|T|\xi_n \rangle = \langle \xi_m|\xi_n \rangle.$$ 

So, $\{\xi_n\}$ is orthonormal implies that $\{U\xi_n\}$ is also orthonormal. □

### 3.3. Trace Class Operators

In this section, we will mainly introduce the representation theorem for trace class operators (see Theorem 3.3.2) and dual relationship between $K(H)$ and $TC(H)$ (see Proposition 3.3.4).

For any pair $(\xi, \eta) \in H \times H$, we define a continuous linear functional $\omega_{\xi, \eta}$ on $B(H)$ by

$$\omega_{\xi, \eta}(T) = \langle T\xi|\eta \rangle \quad (T \in B(H)).$$

(6)

Obviously, $\omega_{\xi, \eta}$ induces a continuous linear functional on $K(H)$ by restriction.

On the other hand, for any $\omega \in K(H)^*$, define $B_\omega : H \times H \longrightarrow \mathbb{C}$ by

$$B_\omega(\xi, \eta) = \langle \xi, \eta \omega \rangle \quad (\xi, \eta \in H).$$
By Proposition 3.2.1, $B_\omega$ is a sesquilinear form on $H$. It is easy to see that $B_\omega$ is bounded. By Theorem 3.1.3, there exists an operator $t(\omega) \in B(H)$ such that

$$\langle t(\omega)\xi|\eta \rangle = B_\omega(\xi, \eta) = \langle t_\xi, \eta \rangle$$

for all $\xi, \eta \in H$. \hspace{1cm} (7)

Note that $\omega \mapsto B_\omega$ is a one-one map, and $B_\omega \mapsto t(\omega)$ is isometric. Thus, $\omega \mapsto t(\omega)$ is also a one-one map. For any $x, y \in H,$

$$\langle t(\omega_\xi, \eta) x|y \rangle = \langle t_{x,y} \xi|\eta \rangle = \langle \langle \xi|y \rangle x|\eta \rangle = \langle \langle x|\eta \rangle \xi|y \rangle = \langle t_{\xi,\eta} x|y \rangle.$$  

Thus, we have

$$t(\omega_\xi, \eta) = t_{\xi, \eta}. \hspace{1cm} (8)$$

**Lemma 3.3.1.** For any $\omega \in K(H)^*$, $t(\omega)$ is a compact operator and

$$\sum_{i \in I} |\langle t(\omega)\xi_i|\xi_i \rangle| < +\infty \hspace{1cm} (9)$$

for every normalized orthogonal system $\{\xi_i\}_{i \in I}$ in $H$.

**Proof.** Let $\omega \in K(H)^*$ and let $\{\xi_i\}_{i \in I}$ be a normalized orthogonal system in $H$. For each $i \in I$, let $\alpha_i$ be a scalar of modulus one such that

$$|\langle t(\omega)\xi_i|\xi_i \rangle| = \alpha_i \langle t(\omega)\xi_i|\xi_i \rangle.$$  

Let $J$ be a finite subset of $I$. Then, since $\|\sum_{i \in J} t_{\alpha_i, \alpha_i}\| = 1$, we have

$$\sum_{i \in J} |\langle t(\omega)\xi_i|\xi_i \rangle| = \sum_{i \in J} \alpha_i \langle t(\omega)\xi_i|\xi_i \rangle = \sum_{i \in J} \langle t_{\alpha_i, \alpha_i}, \omega \rangle = \sum_{i \in J} t_{\alpha_i, \alpha_i}, \omega \leq \|\omega\|.$$  

Therefore, the set $\{i \in I : \langle t(\omega)\xi_i|\xi_i \rangle \neq 0\}$ is countable and series (9) converges.

Now, let $t(\omega) = uh$ be the polar decomposition of $t(\omega)$. We define a continuous functional $\varphi$ on $K(H)$ by $\varphi(x) = \omega(x^*u)$. Then we have

$$\langle t(\varphi)\xi|\eta \rangle = \langle t_{\xi, \eta}, \varphi \rangle = \langle t_{\xi, \eta} u^*, \omega \rangle = \langle t_{\xi, \eta}, \omega \rangle = \langle t(\omega)\xi|\eta \rangle = \langle u^* t(\omega)\xi|\eta \rangle = \langle h\xi|\eta \rangle.$$  

Hence we get $t(\varphi) = h$. Note that $t(\omega)$ is compact if and only if $h = |t(\omega)|$ is compact. Therefore, to prove the compactness of $t(\omega)$, we may assume that $t(\omega)$ is positive. Let

$$t(\omega) = \int_0^{\|t(\omega)\|} \lambda d\mu(\lambda)$$
be the spectral decomposition of \( t(\omega) \). Suppose that a projection

\[
1 - e(\varepsilon) = \int_\varepsilon^1 de(\lambda)
\]

is of infinite rank for some \( \varepsilon > 0 \). Then since

\[
\langle t(\omega)\xi|\xi\rangle = \int_0^\varepsilon \lambda d ||e(\lambda)\xi||^2 \geq \int_\varepsilon^1 \lambda d ||e(\lambda)\xi||^2 \geq \varepsilon ||\xi||^2
\]

for all \( \xi \in (1 - e(\varepsilon))H \), we have

\[
\sum_{n=1}^\infty \langle t(\omega)\xi_n|\xi_n\rangle = \infty
\]

for an infinite normalized orthogonal system \( \{\xi_n\} \) in \((1 - e(\varepsilon))H\). This contradicts the conclusion of our arguments above. Hence, the projection \( 1 - e(\varepsilon) \) is of finite rank for every \( \varepsilon > 0 \). By the inequality

\[
||t(\omega) - (1 - e(\varepsilon))t(\omega)|| = ||e(\varepsilon)t(\omega)|| \leq \varepsilon,
\]

t(\omega) is a limit of operators with finite rank in the norm topology. So \( t(\omega) \) is compact.

\[ \square \]

**Theorem 3.3.2.** Every \( \omega \in K(H)^* \) is of the form

\[
\omega = \sum_n \alpha_n \omega_{\xi_n,\eta_n} \text{ and } ||\omega|| = \sum_n |\alpha_n| \quad (10)
\]

for some normalized orthogonal systems \( \{\xi_n\}, \{\eta_n\} \) and some sequence \( \alpha_n \) in \( l^1 \). Conversely, to any two normalized orthogonal systems \( \{\xi_n\}, \{\eta_n\} \) and any sequence \( \{\alpha_n\} \) in \( l^1 \), there corresponds a unique \( \omega \in K(H)^* \) by equality (10).

**Proof.** Let \( \omega \) be an element of \( K(H)^* \). By Lemma 3.3.1, a compact operator \( t(\omega) \) corresponds to \( \omega \). By Proposition 3.2.2, \( t(\omega) \) is of the form

\[
t(\omega) = \sum_n \alpha_n t_{\eta_n,\xi_n}
\]

for some normalized orthogonal systems \( \{\xi_n\}, \{\eta_n\} \) and some positive sequence \( \alpha = \{\alpha_n\} \) in \( c_0 \).
For every $\beta = \{\beta_n\} \in c_0$, the operator $t_\beta = \sum_n \beta_n t_\xi,\eta_n$ is a compact operator since it is a limit in the norm topology of operators of finite rank. Define $\langle \alpha, \beta \rangle = \omega(t_\beta)$.

Then

$$|\langle \alpha, \beta \rangle| \leq \|\omega\| \|t_\beta\| = \|\omega\| \|\beta\|_\infty.$$ 

On the other hand,

$$\omega(t_\beta) = \omega(\sum_n \beta_n t_\xi,\eta_n) = \sum_n \beta_n \omega(t_\xi,\eta_n) = \sum_n \beta_n (\sum m \alpha_m t_{\xi_m,\eta_m}) \xi_n |\eta_n) = \sum_n \beta_n (\sum m \alpha_m \langle \xi_n, \xi_m \rangle \eta_n) = \sum \alpha_n \beta_n.$$ 

Then we have

$$|\sum \alpha_n \beta_n| = |\omega(t_\beta)| \leq \|\omega\| \|\beta\|_\infty.$$ 

So, we get $\alpha \in l^1 \cong c_0^*$ and $\|\alpha\|_1 \leq \|\omega\|$.

Now, $t(\omega) = \sum_n \alpha_n t_{\xi_n,\eta_n} = \sum m \alpha_m t(\xi_m,\eta_n) = t(\sum \alpha_n \omega_{m,\xi_n}).$ Since the correspondence $\omega \mapsto t(\omega)$ is one-one, we have $\omega = \sum_n \alpha_n \omega_{m,\xi_n}$.

Conversely, given two normalized orthogonal systems $\{\xi_n\}$ and $\{\eta_n\}$ in $H$ and $\alpha = (\alpha_n) \in l^1$, we put

$$\omega_\alpha = \sum \alpha_n \omega_{m,\xi_n}.$$ 

Then we have, for every $x \in K(H)$,

$$|\langle x, \omega_\alpha \rangle| = |\sum \alpha_n \langle x, \omega_{m,\xi_n} \rangle| = |\sum \alpha_n \langle x, \xi_n |\eta_n \rangle|$$

$$\leq \sum |\alpha_n| |\langle x, \xi_n \rangle| |\eta_n \rangle| \leq \|x\| \sum |\alpha_n| = \|x\| \|\alpha\|_1.$$ 

Hence we have $\|\omega_\alpha\| \leq \|\alpha\|_1$. Then $\alpha \in l^1$ implies $\omega_\alpha \in K(H)^*$. 

Now, apply $t(\omega_\alpha)$ to the first part of this proof, we get $\|\omega_\alpha\| \geq \|\alpha\|_1$. Therefore, $\|\omega_\alpha\| = \|\alpha\|_1$. 

**Definition 3.3.3.** We denote by $TC(H)$ the linear space $\{t(\omega) : \omega \in K(H)^*\}$ with the natural linear operations, and define a norm for each $t(\omega)$ by $\|t(\omega)\| = \|\omega\|$. Then $TC(H)$ is a normed space. Each operator in $TC(H)$ is called a nuclear operator or an operator of trace class.

**Proposition 3.3.4.** The correspondence $\omega \mapsto t(\omega)$ from $K(H)^*$ to $TC(H)$ is linear and isometric. For any $\omega \in K(H)^*$, we have $\|t(\omega)\|_{K(H)} \leq \|t(\omega)\|_{TC(H)}$. 

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FURTHER RESULTS ON $K(H)$, $TC(H)$ AND $B(H)$

PROOF. Let $\omega^1, \omega^2 \in K(H)^*$ and $c_1, c_2 \in \mathbb{C}$. Then $c_1 \omega^1 + c_2 \omega^2 \in K(H)^*$. So, $t(c_1 \omega^1 + c_2 \omega^2) \in TC(H)$. Let $\xi, \eta \in H$. We have

$$
\langle t(c_1 \omega^1 + c_2 \omega^2) | \xi | \eta \rangle = \langle t(\xi, \eta, c_1 \omega^1 + c_2 \omega^2) = \langle t(\xi, \eta, c_1 \omega^1) + \langle t(\xi, \eta, c_2 \omega^2) 
= c_1 \langle t(\omega^1) | \xi | \eta \rangle + c_2 \langle t(\omega^2) | \xi | \eta \rangle = \langle ((c_1 t(\omega^1) + c_2 t(\omega^2)) | \xi | \eta \rangle.
$$

Since $\xi, \eta \in H$ are arbitrary, we have

$$
t(c_1 \omega^1 + c_2 \omega^2) = c_1 t(\omega^1) + c_2 t(\omega^2).
$$

From equality (7), we see that

$$
\|t(\omega)\|_{B(H)} = \sup_{\|\xi\|, \|\eta\| \leq 1} |\langle t(\omega) | \xi | \eta \rangle| = \sup_{\|\xi\|, \|\eta\| \leq 1} |\langle t(\xi, \eta, \omega) \rangle| = \sup_{\|\xi\|, \|\eta\| \leq 1} \|t(\xi, \eta, \omega)\| = \omega.
$$

So, $\|t(\omega)\|_{B(H)} \leq \|t(\omega)\|_{TC(H)}$. □

REMARK 3.3.5. Corresponding to the linear space inclusions $l^1 \subseteq c_0 \subseteq l^\infty$ and the Banach space dualities $c_0^* \cong l^1$, $l^\infty \cong l^1$, by combining Theorem 3.2.2, Theorem 3.3.2 and Theorem 3.4.6, we have the linear space inclusions $TC(H) \subseteq K(H) \subseteq B(H)$ and the Banach space dualities $TC(H) \cong K(H)^*$, $B(H) \cong TC(H)^*$.

3.4. Further Results on $K(H)$, $TC(H)$ and $B(H)$

Based on the former results, we will develop some further results on $K(H)$, $TC(H)$ and $B(H)$. Some of these results are cited from Takesaki’s book [25]. All the results are known, but we give detailed proofs. We first prove that $TC(H)$ is an ideal in $B(H)$, and $TC(H)$ is itself a Banach algebra. Then we prove that $B(H)$ is isometric to a space of matrices $\mathcal{M}_I$, where $I$ is an infinite index set (see Theorem 3.4.5). After proving the second conjugate relation between $K(H)$ and $B(H)$, i.e., $K(H)^{**} \cong B(H)$ in Theorem 3.4.6, we further prove that the correspondence $B(H) \rightarrow K(H)^{**}$ not only is multiplicative (Theorem 3.4.7), but also preserves all the $K(H)$-module structures and $B(H)$-module structures on $K(H)^{**}$ and $B(H)$ (see Theorem 3.4.9).

**Proposition 3.4.1.** $TC(H)$ is an ideal in $B(H)$.

**Proof.** Let $x \in K(H)$ and $a \in B(H)$. Since $K(H)$ is an ideal of $B(H)$, both $ax$ and $xa$ are in $K(H)$. So, for $\omega \in TC(H)$, we can define $a\omega, \omega a \in K(H)^*$ by

$$
\langle x, a\omega \rangle = \langle xa, \omega \rangle \quad \text{and} \quad \langle x, \omega a \rangle = \langle ax, \omega \rangle
$$

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3.4. FURTHER RESULTS ON \(K(H), \text{TC}(H)\) AND \(B(H)\)

for all \(x \in K(H)\). For any \(\xi, \eta \in H\), we have

\[
\langle t(a \omega) \xi | \eta \rangle = \langle t_{\xi, \eta} a \omega \rangle = \langle t_{\xi, a^* \eta} \omega \rangle = \langle t(\omega) \xi | a^* \eta \rangle = \langle at(\omega) \xi | \eta \rangle.
\]

Thus, \(t(a \omega) = at(\omega)\). This shows that \(at(\omega) \in \text{TC}(H)\). Similarly, we can prove that \(t(\omega a) = t(\omega) a\) and \(t(\omega) a \in \text{TC}(H)\). Thus, \(\text{TC}(H)\) is an ideal of \(B(H)\).

\[\square\]

We have seen that \(K(H)\) is a Banach \(B(H)\)-module by taking the operator composition as the module actions. With the dual relation, by defining

\[
\langle x, a \omega \rangle = \langle xa, \omega \rangle \quad \text{and} \quad \langle x, \omega a \rangle = \langle ax, \omega \rangle
\]

for all \(x \in K(H), a \in B(H)\) and \(\omega \in K(H)^*\), we see that \(K(H)^*\) is also a Banach \(B(H)\)-module. Then, the identification \(\text{TC}(H) \cong K(H)^*\) induces a Banach \(B(H)\)-module structure on \(\text{TC}(H)\). On the other hand, since \(\text{TC}(H)\) is an ideal of \(B(H)\), and \(\|\cdot\|_{B(H)} \leq \|\cdot\|_{\text{TC}(H)}\) on \(\text{TC}(H)\), \(\text{TC}(H)\) is also a Banach \(B(H)\)-module under the usual operator composition. From the proof of Proposition 3.4.1, we have \(at(\omega) = t(a \omega)\) and \(t(\omega) a = t(\omega a)\) for all \(a \in B(H)\) and \(\omega \in K(H)^*\). The left hands of the two equalities correspond to the \(B(H)\)-module structure on \(\text{TC}(H)\) under operator composition. The right hands correspond to the \(B(H)\)-module structure on \(\text{TC}(H)\) induced from the identification \(\text{TC}(H) \cong K(H)^*\). Thus, the two module structures on \(\text{TC}(H)\) are the same.

**Proposition 3.4.2.** \(\text{TC}(H)\) is a Banach algebra.

**Proof.** From Proposition 3.4.1, we see that if \(\omega_1, \omega_2 \in K(H)^*\), then \(t(\omega_1) \omega_2 \in K(H)^*\). Moreover, \(t(\omega_1) t(\omega_2) = t(t(\omega_1) \omega_2)\). This shows that \(\text{TC}(H)\) is closed under operator composition. So, it is an algebra. Furthermore,

\[
\|t(\omega_1) t(\omega_2)\| = \|t(t(\omega_1) \omega_2)\| = \|t(\omega_1) \omega_2\| \leq \|t(\omega_1)\| \|\omega_2\| = \|t(\omega_1)\|\|t(\omega_2)\|.
\]

Thus, \(\text{TC}(H)\) is a Banach algebra. \[\square\]

We have seen that any Hilbert space \(H\) is isomorphic to \(l^2(J)\) for some set \(J\). To be more clear, let \(\{e_\alpha\}_{\alpha \in J}\) be the orthonormal basis of \(H\). Then \(H\) is isomorphic to \(l^2(J)\) as Hilbert spaces. So, \(B(H)\) can be identified with \(B(l^2(J))\).

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On the other hand, let $I$ be an infinite index set. We denote by $M_I = M_I(\mathbb{C})$ the vector space of all matrices $F = [v_{\alpha \beta}]_{\alpha, \beta \in I}$ with $v_{\alpha \beta} \in \mathbb{C}, \alpha, \beta \in I$. We use $\mathcal{M}_I$ to denote all matrices in $M_I$ with $\sup_{F'} \|F'\| < \infty$, where the supremum is taken over all finite submatrices $F'$ of $F$. It is easy to see that $\mathcal{M}_I$ is a normed space with $\|F\| = \sup\{\|F'\| : F' \text{ is a finite submatrix of } F\}$. We have

**Lemma 3.4.3.** Let $b = [b_{\alpha \beta}] \in \mathcal{M}_I$. Then for any fixed $\alpha_0 \in I$ and $\beta_0 \in I$, the $1 \times I$ submatrix $[b_{\alpha_0 \beta}]_{\beta \in I}$ and $I \times 1$ submatrix $[b_{\alpha \beta_0}]_{\alpha \in I}$ have at most countably many non-zero entries.

**Proof.** Let $n \in \mathbb{N}$ be any natural number. Let $S_n = \{b_{\alpha \beta} : |b_{\alpha \beta}| > 1/n\}$. Then $S = \bigcup_{n=1}^{\infty} S_n$ is the set of all non-zero entries of $(b_{\alpha \beta})_{\beta \in I}$. We show that each $S_n$ is finite, and so $S$ is at most countable.

Suppose that $S_n$ is infinite. We choose an infinite sequence $\{b_{\alpha_0 m} \}_{m \in \mathbb{N}} \subseteq S_n$. For fixed $N \in \mathbb{N}$, we define $x \in \mathbb{C}^N$ by $x_m = e^{-i\theta_m}/\sqrt{N}$, where $\theta_m$ satisfies $b_{\alpha_0 m} = |b_{\alpha_0 \beta}| e^{i\theta_m} \ (1 \leq m \leq N)$. Then $\|x\| = 1$. So,

$$\frac{\sqrt{N}}{n} = \frac{1}{n} \frac{N}{\sqrt{N}} < \frac{\sum_{m=1}^{N} |b_{\alpha_0 \beta}|}{\sqrt{N}} = \|[b_{\alpha_0 \beta}]^1_{1 \leq m \leq N}\|_2 \leq \|[b_{\alpha_0 \beta}]^1_{1 \leq m \leq N}\|_1 \leq \|b\|_{\mathcal{M}_I}.$$ 

Since $N \in \mathbb{N}$ is arbitrary, it contradicts with $\|b\|_{\mathcal{M}_I} < \infty$. Thus, $S_n$ is finite, and $[b_{\alpha \beta}]_{\beta \in I}$ has at most countably many non-zero entries.

Similarly, for $[b_{\alpha \beta}]_{\alpha \in I}$, let $n \in \mathbb{N}$ be any natural number. Let $S_n = \{b_{\alpha \beta} : |b_{\alpha \beta}| > 1/n\}$. Then $S = \bigcup_{n=1}^{\infty} S_n$ is the set of all non-zero entries of $(b_{\alpha \beta})_{\alpha \in I}$. We show that each $S_n$ is finite, and so $S$ is at most countable.

Suppose that $S_n$ is infinite. We choose an infinite sequence $\{b_{\alpha_0 m} \}_{m \in \mathbb{N}} \subseteq S_n$. For fixed $N \in \mathbb{N}$, we take $x = 1$. Then

$$\|[b_{\alpha_0 \beta}]^1_{1 \leq m \leq N}x\| = \left(\sum_{1 \leq m \leq N} |b_{\alpha_0 \beta}|^2 \right)^{1/2} > \frac{\sqrt{N}}{n}.$$ 

On the other hand,

$$\|[b_{\alpha_0 \beta}]^1_{1 \leq m \leq N}x\| \leq \|[b_{\alpha_0 \beta}]^1_{1 \leq m \leq N}\| \leq \|b\|_{\mathcal{M}_I}.$$ 

Thus, $\|b\|_{\mathcal{M}_I} > \sqrt{N}/n$. Since $N \in \mathbb{N}$ is arbitrary, the inequality is impossible. Thus, $S_n$ is finite, and $[b_{\alpha \beta}]_{\beta \in I}$ has at most countably many non-zero entries. 

\[\square\]
3.4. FURTHER RESULTS ON $K(H)$, $TC(H)$ AND $B(H)$

**Lemma 3.4.4.** Let $\{e_\alpha\}_{\alpha \in I}$ be the standard orthonormal basis for $l^2(I)$ and $b = [b_{\alpha \beta}] \in M_I$. Then $\sum_{\beta \in I} b_{\alpha \beta} e_\beta \in l^2(I)$ for all $\alpha \in I$. Let $S(I)$ be the linear span of $\{e_\alpha\}_{\alpha \in I}$. For each $b \in M_I$, we define a linear map $\varphi(b) : S(I) \to l^2(I)$ by $\varphi(b)e_\alpha = \sum_{\beta \in I} b_{\alpha \beta} e_\beta$ ($\alpha \in I$). Then $\varphi(b)x \in l^2(I)$ and $\|\varphi(b)x\| \leq \|b\|_{M_I} \|x\|$ for all $x \in S(I)$. So, $\varphi(b)$ can be extended to a bounded linear map from $l^2(I)$ to $l^2(I)$ satisfying $\|\varphi(b)\| \leq \|b\|$.

**Proof.** By Lemma 3.4.3, for fixed $\alpha \in I$, $[b_{\alpha \beta}]_{\beta \in I}$ has at most countably many non-zero entries, say $b_{\beta_m \alpha}$ ($m \in \mathbb{N}$). So,

$$\left\| \sum_{\beta \in I} b_{\beta_m \alpha} e_\beta \right\|_2 = \left( \sum_{m \in \mathbb{N}} |b_{\beta_m \alpha}|^2 \right)^{1/2}.$$

On the other hand, for any $N \in \mathbb{N}$, $[b_{\beta_m \alpha}]_{1 \leq m \leq N}$ is a finite submatrix of $b$, so, we have $\left\| [b_{\beta_m \alpha}]_{1 \leq m \leq N} \right\|_2 \leq \|b\|_{M_I}$. Taking $x = \sum_{N} b_{\beta_m \alpha} e_\alpha$, we have $\left\| [b_{\beta_m \alpha}]_{1 \leq m \leq N} \right\|_2 \|x\|$, i.e., $\sum_{m=1}^{N} |b_{\beta_m \alpha}|^2 \leq \|b\|_{M_I} (\sum_{m=1}^{N} |b_{\beta_m \alpha}|^2)^{1/2}$. Or, $(\sum_{m=1}^{N} |b_{\beta_m \alpha}|^2)^{1/2} \leq \|b\|_{M_I}$. Let $N \to \infty$, we have

$$\left\| \sum_{\beta \in I} b_{\beta_m \alpha} e_\beta \right\|_2 \leq \left( \sum_{m \in \mathbb{N}} |b_{\beta_m \alpha}|^2 \right)^{1/2} \leq \|b\|_{M_I}.$$

Thus, $\sum_{\beta \in I} b_{\beta_m \alpha} e_\beta \in l^2(I)$ for all $b \in M_I$.

Now, for any $x \in S(I)$, let $x = \sum_{n=1}^{k} x_n e_{\alpha_n}$. By definition,

$$\|\varphi(b)x\| = \|\varphi(b)\| \left\| \sum_{n=1}^{k} x_n e_{\alpha_n} \right\| = \left\| \sum_{n=1}^{k} x_n \varphi(b)e_{\alpha_n} \right\| = \left\| \sum_{n=1}^{k} x_n \sum_{\beta \in I} b_{\beta_{\alpha_n} \alpha_n} e_\beta \right\|.$$

By Lemma 3.4.3, for each $\alpha_n$ ($1 \leq n \leq k$), there are at most countably many non-zero entries, say $\{b_{\beta_m(\alpha_n) \alpha_n} : m(\alpha_n) \in \Lambda_n\}$ with $\Lambda_n$ a subset of $\mathbb{N}$. Then the index set $\{\beta_m(\alpha_n) : m(\alpha_n) \in \Lambda_n, 1 \leq n \leq k\}$ is also countable. By re-arranging the index set with $\{\beta_l : l \in \Lambda\}$ with $\Lambda$ a subset of $\mathbb{N}$, we have

$$\|\varphi(b)x\| = \left\| \sum_{n=1}^{k} x_n \sum_{\beta \in I} b_{\beta_{\alpha_n} \alpha_n} e_\beta \right\| = \left\| \sum_{n=1}^{k} x_n \sum_{m(\alpha_n) \in \Lambda_n} b_{\beta_{m(\alpha_n) \alpha_n} \alpha_n} e_{\beta_{m(\alpha_n) \alpha_n}} \right\|$$

$$= \left\| \sum_{l \in \Lambda} \left( \sum_{n=1}^{k} x_n b_{\beta_{l \alpha_n}} \right) e_{\beta_l} \right\| = \left( \sum_{l \in \Lambda} \left( \sum_{n=1}^{k} x_n b_{\beta_{l \alpha_n}} \right)^2 \right)^{1/2}.$$

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In case that \( A \) is a finite, \([b_{\beta \alpha n}]_{A \times k} \) is a finite submatrix of \( b \in M_I \). So, \(||b_{\beta \alpha n}||_{A \times k} \| \leq \|b||_{M_I} \| x ||\), i.e.,

\[
\left[ \sum_{n=1}^{k} x_n b_{\beta \alpha n} \right]^{2} \leq \|b||_{M_I} \| x ||.
\]

If \( A \) is infinite, we can simply view \( A \) as \( \mathbb{N} \). For any \( N \in \mathbb{N} \), \([b_{\beta \alpha n}]_{N \times k} \) is a submatrix of \( b \in M_I \). Similar to the argument above, we have

\[
\left[ \sum_{n=1}^{k} x_n b_{\beta \alpha n} \right]^{2} \leq \|b||_{M_I} \| x ||.
\]

Let \( N \to \infty \), we have

\[
\left[ \sum_{n=1}^{\infty} x_n b_{\beta \alpha n} \right]^{2} \leq \|b||_{M_I} \| x ||.
\]

Thus, in any case, we have

\[
\|\varphi(b)x || \leq \|b||_{M_I} \| x ||.
\]

Since \( l^2(I) \) is the completion of \( S(I) \) in \( l^2 \)-norm, \( \varphi(b) \) can be extended to a bounded linear map \( \varphi(b) : l^2(I) \to l^2(I) \) satisfying \( \|\varphi(b)\| \leq \|b||_{M_I} \) .

**THEOREM 3.4.5.** For any Hilbert space \( H \), \( B(H) \) is isometric to \( M_I \) for some index set \( I \).

**PROOF.** Let \( \{e_\alpha\}_{\alpha \in I} \) be an orthonormal basis of \( H \). Then \( B(H) \cong B(l^2(I)) \). It suffices to prove that \( B(l^2(I)) \cong M_I \) isometrically. From Lemma 3.4.4, we see that the map \( \varphi : M_I \to B(l^2(I)) \) defined by \( \langle \varphi(b)e_\beta | e_\alpha \rangle = b_{\alpha \beta} \) for all \( b = [b_{\alpha \beta}] \in M_I \) is obviously one-one and \( \|\varphi(b)||_{B(l^2(I))} \leq \|b||_{M_I} \) for all \( b \in M_I \). Now we show that \( \varphi \) is also onto and \( \|\varphi(b)||_{B(l^2(I))} \geq \|b||_{M_I} \).

For each \( b \in B(l^2(I)) \), let \( b = [b_{\alpha \beta}]_{\alpha,\beta \in I} \) with \( b_{\alpha \beta} = \langle Be_\beta | e_\alpha \rangle \). We want to show that \( b \in M_I \). Then it is easy to see that \( \varphi(b) = B \) and \( \varphi \) is onto.

Let \( S \) and \( T \) be finite subsets of \( I \). Let \( b^{ST} \) denote the \( S \times T \) submatrix of \( b \). Consider the diagram

\[
\begin{align*}
l^2(I) & \longrightarrow l^2(I) \\
\uparrow \quad & \quad \downarrow \\
\mathbb{C}^T & \longrightarrow \mathbb{C}^S
\end{align*}
\]
where the first row is the map $B$, the second row is the map $b^{S,T}$, the left column is the inclusion map $i_T$, and the right column is the orthogonal projection $P_S$. It is easy to see that the diagram is commutative. So,

$$
\|b^{S,T}\| = \|i_T \circ B \circ P_S\| \leq \|i_T\| \|B\| \|P_S\| = \|B\|.
$$

Taking supremum on $S, T$, we get $\|a\|_{\mathcal{M}_f} \leq \|B\|$.

Combining with Lemma 3.4.4, we see $\|\varphi(b)\|_{B(\mathcal{H})} = \|b\|_{\mathcal{M}_f}$ for all $b \in \mathcal{M}_f$.

**Theorem 3.4.6.** There is an isometric correspondence $f \leftrightarrow x$ between the second conjugate space $K(H)^{**}$ and $B(H)$ determined by

$$
\langle \omega_{\xi,\eta}, f \rangle = \langle x\xi|\eta \rangle \quad (\xi, \eta \in H).
$$

**Proof.** Let $x \in B(H)$, and $\omega = \sum_{n=1}^{\infty} \alpha_n \omega_{\xi_n, \eta_n} \in K(H)^{*}$ be the same as in Theorem 3.3.2. Define a functional $f_x$ on $K(H)^{*}$ by

$$
\langle \omega, f_x \rangle = \sum \alpha_n \langle x\xi_n|\eta_n \rangle.
$$

By equality (10), we have

$$
|\langle \omega, f_x \rangle| \leq \sum |\alpha_n| \|\langle x\xi_n|\eta_n \rangle\| \leq \|x\| \sum |\alpha_n| = \|x\| \|\omega\|.
$$

So, $f_x$ is continuous on $K(H)^{*}$, and

$$
\|f_x\| \leq \|x\|.
$$

Conversely, for any $f \in K(H)^{**}$, define a sesquilinear form $B_f$ on $H$ by

$$
B_f(\xi, \eta) = \langle \omega_{\xi,\eta}, f \rangle.
$$

We have

$$
|B_f(\xi, \eta)| = |\langle \omega_{\xi,\eta}, f \rangle| \leq \|f\| \|\omega_{\xi,\eta}\| \leq \|f\| \|\xi\| \|\eta\|.
$$

So, $B_f$ is bounded. By Theorem 3.1.3, there exists a unique $x_f \in B(H)$ with $\|B_f\| = \|x_f\|$ such that

$$
\langle x_f\xi|\eta \rangle = B_f(\xi, \eta) = \langle \omega_{\xi,\eta}, f \rangle.
$$

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Replacing $\xi, \eta$ by $\xi_n, \eta_n$, by equalities (12) and (14) we have

$$\langle \omega, f_x \rangle = \sum \alpha_n(x_f, \xi_n | \eta_n) = \sum \alpha_n(\omega_{\xi_n, \eta_n}, f) = (\sum \alpha_n \omega_{\xi_n, \eta_n}, f) = \langle \omega, f \rangle$$

for all $\omega \in K(H)^*$. So, $f_x = f$. Thus, $B(H) \rightarrow K(H)^{**}$, $x \mapsto f_x$ is onto. By equality (14), we have $\|f_x\| \leq \|f\| = \|f_x\|$. Combining with inequality (13), we get $\|f_x\| = \|f\|$. Therefore, $B(H) \rightarrow K(H)^{**}$, $x \mapsto f_x$, is a surjective isometry. □

**Theorem 3.4.7.** The correspondence $B(H) \rightarrow K(H)^{**}$, $x \mapsto f_x$, is multiplicative, where the multiplication in $B(H)$ is the operator composition and the multiplication in $K(H)^{**}$ is the Arens product.

**Proof.** It is well-known that any Hilbert space $H$ is reflexive. By Theorem 2.4.1, $K(H)$ is Arens regular, i.e., the first Arens product and the second Arens product on $K(H)^{**}$ are the same, we use the first Arens product here to prove the theorem. Let $x, y \in B(H)$. We denote by $*$ the Arens product on $K(H)^{**}$. We want to show that $f_x * f_y = f_{xy}$. First, we show that

$$f_y * \omega_{\xi, \eta} = \omega_{y\xi, \eta}$$

for all $y \in B(H)$ and $\xi, \eta \in H$. Finally, we conclude our result.

Let $u', v' \in H$. We have

$$(\omega_{\xi, \eta} * t_{u,v})(t_{u',v'}) = \omega_{\xi, \eta}(t_{u,v}, t_{u',v'}) = \omega_{\xi, \eta}(t_{(u'v')}, u,v')$$

$$= \langle (t(\omega_{\xi, \eta})(u'v')u|v') \rangle = \langle (u'|v)(\omega_{\xi, \eta})u|v' \rangle$$

$$= \langle (u'v)t_{\xi, \eta}u|v' \rangle = \langle (u'|v)(\omega_{\xi, \eta}) \rangle$$

$$= \langle (u\eta)v'|v' \rangle = \langle (u\eta)t_{\xi, \eta}u'|v' \rangle$$

$$= \langle t(\omega_{\xi, \eta})(u'|v')u|v' \rangle = (\omega_{\xi, \eta})(u'|v') = \omega_{\xi, \eta}(u'|v')$$

Since $t_{u',v'}$ is arbitrary, we proved that $\omega_{\xi, \eta} * t_{u,v} = \omega_{(u\eta), \xi,v}$.

Now, we have

$$(f_y * \omega_{\xi, \eta})(t_{u,v}) = f_y(\omega_{\xi, \eta} * t_{u,v}) = f_y(\omega_{(u\eta), \xi,v}) = \langle y(\omega_{\xi, \eta})u|v \rangle = \langle (u\eta)v|y\xi |v' \rangle$$
3.4. FURTHER RESULTS ON $K(H), TC(H)$ AND $B(H)$

$$= (t_{y,\eta} \omega_{\xi,\eta}) u|v| = \omega_{y,\xi,\eta} (t_u v).$$

Since $t_{u,v}$ is arbitrary, we proved $f_x \ast \omega_{\xi,\eta} = \omega_{y,\xi,\eta}$. Therefore,

$$f_x \ast f_y (\omega_{\xi,\eta}) = f_x (f_y \ast \omega_{\xi,\eta}) = f_x (\omega_{y,\xi,\eta}) = \langle xy \xi | \eta \rangle = f_{xy} (\omega_{\xi,\eta}).$$

Since $\omega_{\xi,\eta}$ is arbitrary, we have $f_{xy} = f_x \ast f_y$. □

On one hand, $B(H)$ is a Banach $TC(H)$-module. The module actions of $TC(H)$ on $B(H)$ is the usual operator composition.

On the other hand, since $K(H)$ is an ideal of $B(H)$, $B(H)K(H) \subseteq K(H)$ and $K(H)B(H) \subseteq K(H)$. So, $TC(H)K(H) \subseteq K(H)$ and $K(H)TC(H) \subseteq K(H)$. That means $K(H)$ is a $TC(H)$-submodule of $B(H)$. Then the dual relation induces a $TC(H)$-bimodule structure on $K(H)^*$ by

$$\langle x, aw \rangle = \langle xa, \omega \rangle \text{ and } \langle x, \omega a \rangle = \langle ax, \omega \rangle \quad (15)$$

for all $a \in TC(H), x \in K(H)$ and $\omega \in K(H)^*$.

Using the similar way, the $TC(H)$-bimodule structure on $K(H)^*$ induces a bimodule structure on $K(H)^{**}$ by

$$\langle \omega, af \rangle = \langle \omega a, f \rangle \text{ and } \langle \omega, fa \rangle = \langle \omega, af \rangle \quad (16)$$

for all $a \in TC(H), f \in K(H)^{**}$ and $\omega \in K(H)^*$. We claim

**Proposition 3.4.8.** The correspondence between $B(H)$ and $K(H)^{**}$ preserves the above module structures.

**Proof.** We use $\pi$ to denote the map $B(H) \rightarrow K(H)^{**} : x \mapsto f_x$ defined in Theorem 3.4.6. Then the theorem says that $\langle \omega_{\xi,\eta}; \pi(x) \rangle = \langle x \xi | \eta \rangle$ for all $\xi, \eta \in H$ and $x \in B(H)$. We want to show that $\pi(t(\omega')x) = t(\omega') \pi(x)$ for all $x \in B(H)$ and $\omega' \in K(H)^*$. 

Note that, for any $\xi, \eta \in H, t(\omega') \in TC(H)$ and $s \in K(H)$, by equality (15), we have

$$\langle s, \omega_{\xi,\eta} t(\omega') \rangle = \langle t(\omega') s, \omega_{\xi,\eta} \rangle = \langle t(\omega') s \xi | \eta \rangle = \langle s \xi | t(\omega') \eta \rangle = \langle s, \omega_{\xi,t(\omega') \eta} \rangle.$$
Thus, $\omega_{\xi,\eta}t(\omega') = \omega_{\xi,t(\omega')}\eta$. Then, for all $x \in B(H)$,
\[
\langle \omega_{\xi,\eta}t(\omega'), \pi(x) \rangle = \langle \omega_{\xi,t(\omega')}\eta, \pi(x) \rangle = \langle \omega_{\xi,t(\omega')}, \pi(t(\omega')x) \rangle.
\]

Since $\xi, \eta \in H$ are arbitrary, we have $\pi(t(\omega')x) = t(\omega')\pi(x)$ for all $t(\omega') \in TC(H)$ and $x \in B(H)$. Similarly, we also have $\pi(xt(\omega')) = \pi(x)t(\omega')$. Thus, the map $\pi$ preserves the $TC(H)$-bimodule structures. \qed

Since $B(H)$ is an algebra, $B(H)$ is of course a $B(H)$-module. With restriction on $K(H)$, we see that $K(H)$ is a $B(H)$-submodule since $K(H)$ is an ideal of $B(H)$. Using the dual relations similar to above, we see that $K(H)^{**}$ is also a $B(H)$-module. Still using the fact that $K(H)$ is an ideal of $B(H)$, we can see that both $B(H)$ and $K(H)^{**}$ are $K(H)$-modules. We claim

**Proposition 3.4.9.** The map $x \mapsto f_x$ preserves all the $K(H)$-module structures and $B(H)$-module structures on $K(H)^{**}$ and $B(H)$.

**Proof.** Since the proofs are similar, we only show it for the case of $K(H)$.

Let $a \in K(H), x \in B(H)$, and $f_x \in K(H)^{**}$ the correspondence of $x$ as defined in Theorem 3.4.6. Let $\xi, \eta \in H$ and $\omega_{\xi,\eta} \in K(H)^*$ be defined as in Equality (6). Then, from the proof of Proposition 3.4.1, we see that
\[
t(\omega_{\xi,\eta}a) = t(\omega_{\xi,\eta}a) = t_{\xi,\eta} = t(\omega_{\xi,\eta}).
\]

Thus, $\omega_{\xi,\eta}a = \omega_{\xi,\eta}a$ for all $a \in K(H)$ and $\xi, \eta \in H$. Now,
\[
\langle \omega_{\xi,\eta}a, f_x \rangle = \langle \omega_{\xi,\eta}a, f_x \rangle = \langle \omega_{\xi,\eta}a, f_x \rangle = \langle \omega_{\xi,\eta}a, f_x \rangle = \langle ax\xi | \eta \rangle = \langle ax\xi | \eta \rangle = \langle ax\xi | \eta \rangle.
\]

Therefore, $af_x = f_{ax}$. This shows that the map $x \mapsto f_x$ preserves the left module structures on $K(H)^{**}$ and $B(H)$. The case of right module can be similarly proved. \qed
CHAPTER 4

Introduction to Operator Spaces

This chapter is an introduction to the concept of operator space and its fundamental properties, which is based on the monograph by E. G. Effros and Z.-J. Ruan (see [13]). We start with the definition of concrete operator space. Then we introduce abstract operator spaces and prove that any concrete operator space is an abstract operator space. The inverse is also true, i.e., every abstract operator space has a representation of concrete operator space over some Hilbert space, which is proved in Section 4.3. In the rest of this chapter, we discuss some important definitions and properties for operator spaces.

4.1. Concrete Operator Spaces and Abstract Operator Spaces

In this section, the concepts of concrete operator space and abstract operator space are introduced. It is proved that every concrete operator space is an abstract operator space. We cite those definitions from [13].

**Definition 4.1.1.** A concrete operator space on Hilbert space $H$ is a linear subspace $V$ of $B(H)$.

**Definition 4.1.2.** A matrix norm $\| \cdot \|$ on a linear space $V$ is a sequence of norms on the matrix space $M_n(V)$ for each $n \in \mathbb{N}$. We denote by $M_n(V)$ the normed space $\| \cdot \|_{M_n(V)}$.

Let $V$ be a vector space, and $v \in M_m(V)$ and $w \in M_n(V)$, $(m, n \in \mathbb{N})$. The direct sum of $v$ and $w$, denoted by $v \oplus w$, is defined to be an element in $M_{m+n}(V)$ with a form

\[
\begin{bmatrix}
v & 0_{m,n} \\
0_{n,m} & w
\end{bmatrix}.
\]

**Definition 4.1.3.** An abstract operator space is a linear space $V$ together with a matrix norm $\| \cdot \|$ satisfying

- $M1$ \[ \|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\} \]
• $M_2$ \[ \|\alpha \psi \beta \|_n \leq \|\alpha\|_m \|\beta\| \]

for all $\alpha \in M_m(V), \psi \in M_n(V)$, and $\beta \in M_{m,n}$, where the norms of $\alpha$ and $\beta$ are operator norms, i.e., we treat $\alpha$ and $\beta$ as elements of $B(C^m, C^n)$ and $B(C^n, C^m)$, respectively.

If $V$ is an operator space, then we call the matrix norm an operator space structure.

Note that Condition $M_1$ can be replaced by the following

• $M_1'$ \[ \|\psi \otimes \omega\|_{m+n} \leq \max\{\|\psi\|_m, \|\omega\|_n\}. \]

This can be easily seen from Condition $M_2$ since we have

\[ \|\psi \| = \|\begin{bmatrix} I_m & 0_{m,n} \\ 0_{n,m} & I_n \end{bmatrix} (\psi \otimes \omega) \| \leq \|\psi \otimes \omega\| \]

for $\psi \in M_m(V)$ and $\omega \in M_n(V)$. Similarly, we have $\|\omega\| \leq \|\psi \otimes \omega\|$ for $\omega \in M_n(V)$.

Combining with $M_1'$, we get $M_1$.

**Lemma 4.1.4.** Let $H_1, H_2, K_1$, and $K_2$ be Hilbert spaces. Let $b_1 : H_1 \to K_1$ and $b_2 : H_2 \to K_2$ be two linear operators. Define $b_1 \otimes b_2$ to be an operator from $H_1 \otimes H_2$ to $K_1 \otimes K_2$ determined by

\[ (b_1 \otimes b_2)(h_1 \otimes h_2) = (b_1 h_1) \otimes (b_2 h_2) \]

for all $h_1 \in H_1$ and $h_2 \in H_2$. Then $\|b_1 \otimes b_2\| = \|b_1\| \|b_2\|$. 

**Proof.** First we have $\|b_1 \otimes b_2\| \geq \|b_1\| \|b_2\|$. This is because

\[ \|b_1 \otimes b_2\| = \sup \{\|b_1 \otimes b_2(h)\| : h \in H_1 \otimes H_2, \|h\| \leq 1\} \]

\[ \geq \sup \{\|b_1 \otimes b_2(h_1 \otimes h_2)\| : h_1 \in H_1, h_2 \in H_2, \|h_1 \otimes h_2\| \leq 1\} \]

\[ = \sup \{\|b_1 h_1 \otimes b_2 h_2\| : h_1 \in H_1, h_2 \in H_2, \|h_1 \otimes h_2\| \leq 1\} \]

\[ \geq \sup \{\|b_1 h_1\| \|b_2 h_2\| : h_1 \in H_1, h_2 \in H_2, \|h_1\| \leq 1, \|h_2\| \leq 1\} \]

\[ = \sup \{\|b_1 h_1\| \|b_2 h_2\| : h_1 \in H_1, h_2 \in H_2, \|h_1\| \leq 1, \|h_2\| \leq 1\} \]

\[ = \|b_1\| \|b_2\|. \]

So, it suffices to show that $\|b_1 \otimes b_2\| \leq \|b_1\| \|b_2\|$.

Let $f(x_1, y_1) = \|b_1\|^2 (x_1, y_1) - (b_1 x_1, b_1 y_1)$ and $\varphi(x_1, y_1; x_2, y_2) = f(x_1, y_1)(x_2, y_2)$ for $x_1, y_1 \in H_1, x_2, y_2 \in H_2$. Since $f(x, x) = \|b_1\|^2 \|x\|^2 - \|b_1 x\|^2 \geq 0$ for all $x \in H$, we
have

\[ \varphi(x, y; x, y) = f(x, x)\langle y, y \rangle \geq 0. \quad (17) \]

On the other hand,

\[ \varphi(x_1, y_1; x_2, y_2) = (\|b_1\|^2\langle x_1, y_1 \rangle - \langle b_1x_1, b_1y_1 \rangle)\langle x_2, y_2 \rangle \]

\[ = \|b_1\|^2\langle x_1, y_1 \rangle\langle x_2, y_2 \rangle - \langle b_1x_1, b_1y_1 \rangle\langle x_2, y_2 \rangle \]

\[ = \|b_1\|^2\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle - \langle b_1x_1 \otimes x_2, b_1y_1 \otimes y_2 \rangle \]

\[ = \|b_1\|^2\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle - \langle (b_1 \otimes I)(x_1 \otimes x_2), (b_1 \otimes I)(y_1 \otimes y_2) \rangle. \]

Now, we can rewrite \( \varphi(x_1, y_1; x_2, y_2) \) as \( \varphi(x_1 \otimes x_2, y_1 \otimes y_2) \) since the above expression only contains variables with the form \( u \otimes v \). Then we extend it linearly to the whole linear space \( H_1 \otimes H_2 \). We have

\[ \phi(x_1, x_2) = \|b_1\|^2\langle x_1, x_2 \rangle - \langle (b_1 \otimes I)x_1, (b_1 \otimes I)x_2 \rangle. \quad (18) \]

We want to prove that \( \varphi(z, z) \geq 0 \) for all \( z \in H_1 \otimes H_2 \).

By definition, for any \( x_1, y_1 \in H_1 \) and \( x_2, y_2 \in H_2 \), we have

\[ \varphi(x_1 \otimes x_2, y_1 \otimes y_2) = \|b_1\|^2\langle x_1, y_1 \rangle\langle x_2, y_2 \rangle - \langle b_1x_1, b_1y_1 \rangle\langle x_2, y_2 \rangle \]

\[ = (\|b_1\|^2\langle x_1, y_1 \rangle - \langle b_1x_1, b_1y_1 \rangle)\langle x_2, y_2 \rangle. \]

For any \( z = \sum_{i=1}^{m} x_i \otimes y_i \in H_1 \otimes H_2 \), let \( X = \vee \{y_i : i = 1, \ldots, m\} \), the linear span of \( \{y_i : i = 1, \ldots, m\} \), and choose an orthogonal basis of \( X \), say \( \{e_i : i = 1, \ldots, n\} \) \( (n \leq m) \).

Then \( z = \sum_{i=1}^{n} \tilde{x}_i \otimes e_i \), for some \( \tilde{x}_i \in H_1 \). Now,

\[ \varphi(z, z) = \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(\tilde{x}_i \otimes e_i, \tilde{x}_j \otimes e_j) = \sum_{i=1}^{n} \varphi(\tilde{x}_i \otimes e_i, \tilde{x}_i \otimes e_i) \geq 0 \]

since each \( \varphi(\tilde{x}_i \otimes e_i, \tilde{x}_i \otimes e_i) \geq 0 \) (see Equality (17)). By Equality (18), \( \varphi(z, z) \geq 0 \) implies

\[ \varphi(z, z) = \|b_1\|^2\|z\|^2 - \|(b_1 \otimes I)z\|^2 \geq 0. \]

Thus, \( \|(b_1 \otimes I)z\| \leq \|b_1\|\|z\| \) for all \( z \in H_1 \otimes H_2 \), from which it is immediately that

\[ \|b_1 \otimes I\| \leq \|b_1\|. \]
By the same way, we have \( \| I \otimes b_2 \| \leq \| b_2 \| \). Since \( b_1 \otimes b_2 \) can be viewed as a composition of \( b_1 \otimes I \) and \( I \otimes b_2 \), it follows that

\[
\| b_1 \otimes b_2 \| = \|(b_1 \otimes I) \circ (I \otimes b_2)\| \leq \| b_1 \otimes I \| \| I \otimes b_2 \| \leq \| b_1 \| \| b_2 \|.
\]

Therefore, \( \| b_1 \otimes b_2 \| = \| b_1 \| \| b_2 \| \).

**Proposition 4.1.5.** Let \( V \) be an abstract operator space. Then for any matrix \( v \in M_n(V) \) and \( \alpha \in M_p \), we have

\[
\| v \otimes \alpha \| = \| \alpha \otimes v \| = \| v \| \| \alpha \|.
\]

**Proof.** By Polar Decomposition Theorem, we may assume \( \alpha = \mu \mid \alpha \mid \), where \( \mu \) is unitary. From the finite-dimensional spectral theorem, there is a unitary matrix \( \lambda \in M_p \), and scalars \( c_1 \geq c_2 \geq \ldots \geq c_p \geq 0 \) such that \( \| \alpha \| = c_1 \) and \( \| \mid \alpha \mid \| = \lambda^* (c_1 \oplus \ldots \oplus c_p) \lambda \).

If we let \( \tilde{v} = (c_1 \oplus \ldots \oplus c_p) \otimes v = c_1 v \oplus \ldots \oplus c_p v \in M_{pn}(V) \), then by M1,

\[
\| \alpha \otimes v \| = \| \mu \lambda^* (c_1 \oplus \ldots \oplus c_p) \lambda \otimes v \| = \| (\mu \lambda^* \otimes I_n) \tilde{v} (\lambda \otimes I_n) \| = \| \tilde{v} \| = \| \alpha \| \| v \|.
\]

Since \( \alpha \otimes v = v \otimes \alpha \), we are done.

**Proposition 4.1.6.** Let \( V \) be an operator space with norm \( \| \cdot \| \). Then for any \( v = [v_{i,j}] \in M_n(V) \), \( \| v_{i,j} \| \leq \| v \| \leq \sum_{1 \leq i,j \leq n} \| v_{i,j} \| \), where \( n \in \mathbb{N} \) and \( 1 \leq i, j \leq n \).

**Proof.** By operator space condition M2, we have

\[
\| v_{i,j} \| = \| E_i^* v E_j \| \leq \| E_i \| \| v \| \| E_j^* \| = \| v \|
\]

\[
= \| \sum_{1 \leq i,j \leq n} E_i v_{i,j} E_j^* \| \leq \sum_{1 \leq i,j \leq n} \| E_i v_{i,j} E_j^* \| \leq \sum_{1 \leq i,j \leq n} \| v_{i,j} \|.
\]

**Proposition 4.1.7.** Every concrete operator space is an abstract operator space.

Let \( H \) be a Hilbert space, \( V \subseteq B(H) \) a linear subspace of \( B(H) \). We want to show that, for all \( v \in M_m(V), w \in M_n(V), \alpha \in M_{n,m}, \) and \( \beta \in M_{m,n} \) with \( m, n \in \mathbb{N} \), the conditions M1' and M2 are satisfied with a suitable assignment of a matrix norm. Before we present the proof, we need the following lemmas.
4.1. CONCRETE OPERATOR SPACES AND ABSTRACT OPERATOR SPACES

LEMMA 4.1.8. We have a linear isomorphism

$$\mathcal{M}_n(B(H)) \cong B(H^n).$$

So we can use the norm on $B(H^n)$ to define a norm on $\mathcal{M}_n(B(H))$. Then $B(H)$ has a matrix norm.

PROOF. Any $b = [b_{i,j}]_{n \times n} \in \mathcal{M}_n(B(H))$ determines an operator on $H^n$ defined by

$$\varphi_n(b)(\eta) = (\sum_{i,j} b_{i,j} \eta_{i,j}, \ldots, \sum_{i,j} b_{i,j} \eta_{i,j})$$

for all $\eta = (\eta_1, \ldots, \eta_n) \in H^n$. We claim that $\varphi_n$ is an isomorphism from $\mathcal{M}_n(B(H))$ onto $B(H^n)$ for each $n \in \mathbb{N}$. This induces a matrix norm on $B(H)$.

To show that $\varphi_n$ is onto, define $\psi_i : H \rightarrow H^n$ by $\psi_i(h) = (0, \ldots, 0, h, 0, \ldots, 0)$ for all $h \in H$ and $1 \leq i \leq n$. Then $\psi_i$ is an isomorphism from $H$ onto $H_i = \{(0, \ldots, 0, h, 0, \ldots, 0) : h \in H\}$ for each $1 \leq i \leq n$. For any $T \in B(H^n)$ and $h = (h_1, \ldots, h_n) \in H^n$, we have $Th = T(\sum_{i=1}^n \tilde{h}_i) = \sum_{i=1}^n T\tilde{h}_i$, where $\tilde{h}_i = \psi_i(h_i)$. Since $T\tilde{h}_i \in H^n$, $T\tilde{h}_i = (T_1 \tilde{h}_i, \ldots, T_n \tilde{h}_i) = (b_{i_1}h_i, \ldots, b_{i_n}h_i)$, where $b_{ij} = T_j \circ \psi_i$ ($1 \leq j \leq n$). Then

$$Th = \sum_{i=1}^n T\tilde{h}_i = (\sum_{i=1}^n T_1 \tilde{h}_i, \ldots, \sum_{i=1}^n T_n \tilde{h}_i) = (\sum_{i=1}^n b_{i_1}h_i, \ldots, \sum_{i=1}^n b_{i_n}h_i) = \varphi_n(b)(h).$$

\[ \square \]

LEMMA 4.1.9. $H^n \cong \mathbb{C}^n \otimes_2 H$ in the sense of Hilbert space isomorphism.

PROOF. Let $h = (h_1, h_2, \ldots, h_n) \in H^n$ and $E_i$ ($i = 1, \ldots, n$) be the vector units, i.e., $E_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Define $\varphi : H^n \rightarrow \mathbb{C}^n \otimes_2 H$ by $\varphi(h) = \sum_{i=1}^n E_i \otimes h_i$.

We show that $\varphi$ is an isomorphism.

First, given $h, k \in H$, and $\alpha, \beta \in \mathbb{C}$, we have

$$\varphi(\alpha h + \beta k) = \varphi(\alpha h_1 + \beta k_1, \ldots, \alpha h_n + \beta k_n) = \sum_{i=1}^n E_i \otimes (\alpha h_i + \beta k_i)$$

$$= \sum_{i=1}^n (\alpha E_i \otimes h_i + \beta E_i \otimes k_i) = \alpha \sum_{i=1}^n E_i \otimes h_i + \beta \sum_{i=1}^n E_i \otimes k_i$$

$$= \alpha \varphi(h) + \beta \varphi(k).$$

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So, \( \varphi \) is linear.

Second, for any \( h = (h_1, h_2, \ldots, h_n) \in H^n \), \( \varphi(h) = 0 \) if and only if \( \sum_{i=1}^{n} E_i \otimes h_i = 0 \) if and only if \( E_i \otimes h_i = 0 \) for all \( 1 \leq i \leq n \) if and only if \( h_i = 0 \) for all \( 1 \leq i \leq n \), i.e., \( h = 0 \). So, \( \varphi \) is injective.

Third, for \( x \in \mathbb{C}^n \otimes H \), \( x \) has a representation \( \sum_{i=1}^{m} \alpha_i \otimes h_i \), where \( \alpha_i \in \mathbb{C}^n \) and \( h_i \in H \) \( (1 \leq i \leq m) \). For each \( \alpha_i \in \mathbb{C}^n \), \( \alpha_i \) has a representation \( \alpha_i = \sum_{j=1}^{n} \alpha_{i,j} E_j \). So

\[
x = \sum_{i=1}^{m} \alpha_i \otimes h_i = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \alpha_{i,j} E_j \right) \otimes h_i = \sum_{j=1}^{m} \left( \sum_{i=1}^{m} \alpha_{i,j} E_j \right) \otimes h_i
\]

Let \( k_j = \sum_{i=1}^{m} \alpha_{i,j} h_i \) and \( k = (k_1, \ldots, k_n) \). Then \( k \in H^n \) and \( \varphi(k) = x \). Thus, \( \varphi \) is surjective from \( H^n \) onto the algebraic tensor product space \( \mathbb{C}^n \otimes H \). This proved that \( \varphi \) is an isomorphism from \( H^n \) onto the algebraic space \( \mathbb{C}^n \otimes H \).

Fourth, for \( h_1, h_2 \in H^n \), we have, on the one hand,

\[
\langle h_1, h_2 \rangle = \langle (h_1, \ldots, h_n), (h_1, \ldots, h_n) \rangle = \langle h_1, h_1 \rangle + \langle h_2, h_2 \rangle + \ldots + \langle h_n, h_n \rangle = \sum_{i=1}^{n} \langle h_i, h_i \rangle,
\]

and on the other hand,

\[
\langle \varphi(h_1), \varphi(h_2) \rangle = \langle \sum_{i=1}^{n} E_i \otimes h_1, \sum_{j=1}^{n} E_j \otimes h_2 \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle E_i \otimes h_1, E_j \otimes h_2 \rangle
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle E_i, E_j \rangle \langle h_1, h_2 \rangle = \sum_{i=1}^{n} \langle h_1, h_2 \rangle = \langle h_1, h_2 \rangle.
\]

Thus, \( \varphi \) is an isometry. \( \square \)

**Lemma 4.1.10.** Let \( \alpha \in M_{n,m}, \beta \in M_{m,n} \) and \( v \in M_{n}(\mathcal{B}(H)) \). Then \( \alpha v \beta = (\alpha \otimes I)v(\beta \otimes I) \) under the identification \( H^n \cong \mathbb{C}^n \otimes H \).

**Proof.** For any \( h = (h_1, \ldots, h_n) \in H^n \), with identification \( H^n \cong \mathbb{C}^n \otimes H \), we want to prove

\[
(\alpha \otimes I)v(\beta \otimes I)(\sum_{i=1}^{n} E_i \otimes h_i) = (\alpha v \beta)(h_1, \ldots, h_n),
\]

where \( E_i \) are unit vectors of \( \mathbb{C}^n \) whose \( i \)-th element is 1 with others zero.

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First, we have

\[ (\beta \otimes I)(\sum_{i=1}^{n} E_i \otimes h_i) = \sum_{i=1}^{n} (\beta \otimes I)(E_i \otimes h_i) = \sum_{i=1}^{n} (\beta E_i) \otimes (Ih_i) \]

\[ = \sum_{i=1}^{n} (\beta E_i) \otimes h_i = \sum_{j=1}^{n} (\sum_{i=1}^{m} \beta_{i,j} E_j) \otimes h_i \]

\[ = \sum_{j=1}^{m} \sum_{i=1}^{n} (\beta_{i,j} E_j) \otimes h_i = \sum_{j=1}^{m} E_j \otimes (\sum_{i=1}^{n} \beta_{i,j} h_i) \]

\[ = (\sum_{i=1}^{n} \beta_{i,1} h_i, \ldots, \sum_{i=1}^{n} \beta_{i,m} h_i) = \beta h. \]

Then,

\[ v(\beta \otimes I)(\sum_{i=1}^{n} E_i \otimes h_i) = v(\sum_{i=1}^{n} \beta_{i,1} h_i, \ldots, \sum_{i=1}^{n} \beta_{i,m} h_i) \]

\[ = (\sum_{i=1}^{n} \sum_{j=1}^{m} (\sum_{i=1}^{m} \beta_{i,j}) h_i, \ldots, \sum_{i=1}^{n} \sum_{j=1}^{m} (\sum_{j=1}^{m} \beta_{i,j}) h_i) \]

\[ = (v\beta)h = \sum_{k=1}^{n} E_k \otimes \tilde{h}_k, \]

where \( \tilde{h}_k = \sum_{i=1}^{n} \sum_{j=1}^{m} (\sum_{i=1}^{m} \beta_{i,j}) h_i. \)

Finally, with the first two steps, we have

\[ (\alpha \otimes I)v(\beta \otimes I)(\sum_{i=1}^{n} E_i \otimes h_i) \]

\[ = (\alpha \otimes I)(\sum_{k=1}^{m} E_k \otimes \tilde{h}_k) = (\sum_{k=1}^{m} \alpha_{1,k} \tilde{h}_k, \ldots, \sum_{k=1}^{m} \alpha_{n,k} \tilde{h}_k) \]

\[ = (\sum_{k=1}^{m} \alpha_{1,k} (\sum_{i=1}^{m} (\sum_{j=1}^{m} \beta_{i,j}) h_i), \ldots, (\sum_{k=1}^{m} \alpha_{n,k} (\sum_{j=1}^{m} (\sum_{i=1}^{m} \beta_{i,j}) h_i)) \]

\[ = (\sum_{k=1}^{m} (\sum_{i=1}^{m} (\sum_{j=1}^{m} \alpha_{1,k} \beta_{i,j}) h_i), \ldots, (\sum_{k=1}^{m} (\sum_{j=1}^{m} (\sum_{i=1}^{m} \alpha_{n,k} \beta_{i,j}) h_i)) \]

\[ = (\alpha v \beta)h. \]

Still with the identification of \( h = (h_1, \ldots, h_n) = \sum_{i=1}^{n} E_i \otimes h_i, \) we therefore have

\[ \alpha v \beta = (\alpha \otimes I)v(\beta \otimes I). \]
Now we prove Proposition 4.1.7.

**Proof.** Let \( v \in M_m(V), w \in M_n(V) \), and let \( \varphi^n \) be the isomorphism from \( M_n(B(H)) \) to \( B(H^n) \). Then

\[
\|v \otimes w\|_{m+n} = \left\| \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \right\|_{m+n} = \left\| \varphi^{m+n} \left( \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \right) \right\|_{B(H^{m+n})}
\]

\[
= \sup \left\{ \left\| \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\| : x \in H^m, y \in H^n, \|x\|^2 + \|y\|^2 \leq 1 \right\}
\]

\[
= \sup \{\|vx, wy\| : x \in H^m, y \in H^n, \|x\|^2 + \|y\|^2 \leq 1 \}
\]

\[
= \sup \{\|vx\|^2 + \|wy\|^2 : x \in H^m, y \in H^n, \|x\|^2 + \|y\|^2 \leq 1 \}
\]

\[
\leq \sup \{\|v\|^2\|x\|^2 + \|w\|^2\|y\|^2 : x \in H^m, y \in H^n, \|x\|^2 + \|y\|^2 \leq 1 \}
\]

\[
\leq \max\{\|v\|_m, \|w\|_n\} \sup \{\|v\|^2 + \|w\|^2 : x \in H^m, y \in H^n, \|x\|^2 + \|y\|^2 \leq 1 \}
\]

\[
= \max\{\|v\|_m, \|w\|_n\}.
\]

This proved \( M1' \). On the other hand,

\[
\|\alpha u \beta\| = \|(\alpha \otimes I)v(\beta \otimes I)\| \leq \|\alpha \otimes I\|\|v\|\|\beta \otimes I\| = \|\alpha\|\|v\|\|\beta\|.
\]

This proved \( M2 \). \qed

### 4.2. Completely Bounded Linear Mappings

Let \( V \) and \( W \) be two abstract operator spaces and \( \varphi : V \rightarrow W \) a linear mapping. For each \( n \in \mathbb{N} \), define \( \varphi^{(n)} : M_n(V) \rightarrow M_n(W) \) by

\[
\varphi^{(n)}(v) = [\varphi(v_{ij})]
\]

for all \( v = [v_{ij}] \in M_n(V) \). Then \( \varphi^{(n)} \) is also a linear mapping, called the \( n \)-th amplification of \( \varphi \). We say that \( \varphi \) is completely bounded if \( \sup_{n \in \mathbb{N}} \|\varphi^{(n)}\| < \infty \). Let \( CB(V, W) \) denote the set of all such \( \varphi \in B(V, W) \). Then \( CB(V, W) \) is a normed space with the complete bounded norm of \( \varphi \) defined by

\[
\|\varphi\|_{cb} = \sup_{n \in \mathbb{N}} \|\varphi^{(n)}\|.
\]
4.2. COMPLETELY BOUNDED LINEAR MAPPINGS

We say that \( \varphi \) is completely isometric (respectively, completely contractive) if all \( \varphi^{(n)} \) are isometric (respectively, contractive).

Given two normed spaces \( V \) and \( W \), a bounded linear mapping \( \varphi : V \rightarrow W \) is said to be a quotient mapping if the induced mapping \( \bar{\varphi} : V/\text{ker}(\varphi) \rightarrow W \) is a surjective isometry. \( \varphi \) is called an exact quotient mapping if \( \bar{\varphi} \) maps \( V/\text{ker}(\varphi) \) onto \( W \). \( \varphi \) is called a complete quotient mapping if each \( \varphi^{(n)} \) is a quotient mapping. \( \varphi \) is called an exact complete quotient mapping if each \( \varphi^{(n)} \) is an exact quotient mapping. Given an abstract operator space \( V \) and a Hilbert space \( H \), we say that a mapping \( \varphi : V \rightarrow B(H) \) is a realization of \( V \) if it is a completely isometric injection.

**Proposition 4.2.1.** \( \|\varphi\| \leq \|\varphi^{(2)}\| \leq \cdots \leq \|\varphi^{(n)}\| \leq \|\varphi\|_{cb} \).

**Proof.** This can be easily seen from the following inequality.

\[
\|\varphi^{(m+n)}\| = \sup_{\|v\| \leq 1} \|\varphi^{(m+n)}(v)\| \geq \sup_{\|v_m \| \leq 1} \|\varphi^{(m+n)}(v_m \oplus v_n)\| \\
= \sup_{\max(\|v_m\|,\|v_n\|) \leq 1} \|\varphi^{(m)}(v_m) \oplus \varphi^{(n)}(v_n)\| \\
\geq \sup_{\|v_m\| \leq 1} \|\varphi^{(m)}(v_m)\| = \|\varphi^{(m)}\|
\]

\[\square\]

**Lemma 4.2.2.** Given \( m, n \in \mathbb{N} \) with \( m \geq n \), and a vector \( \eta \in \mathbb{C}^m \otimes \mathbb{C}^n \), there exists an isometry \( \beta : \mathbb{C}^n \rightarrow \mathbb{C}^m \) and a vector \( \tilde{\eta} \in \mathbb{C}^m \otimes \mathbb{C}^n \) such that \( (\beta \otimes I_n)(\tilde{\eta}) = \eta \)

**Proof.** Let \( \{\varepsilon_j^{(n)} : j = 1, 2, ..., n\} \) be a basis of \( \mathbb{C}^n \). For any \( \eta \in \mathbb{C}^m \otimes \mathbb{C}^n \), there exists a unique decomposition with \( \eta_j \in \mathbb{C}^m (j = 1, 2, ..., n) \) such that

\[
\eta = \sum_{j=1}^{n} \eta_j \otimes \varepsilon_j^{(n)} \quad (21)
\]

Let \( F \) be the subspace spanned by \( \{\eta_j : j = 1, ..., n\} \). Then \( \dim(F) \leq n \leq m \). Thus, there is an isometry \( \beta : \mathbb{C}^{\dim(F)} \rightarrow \mathbb{C}^m \). If \( \dim(F) < n \), we can extend \( \beta \) to \( \mathbb{C}^n \) isometrically. So we can always denote it by \( \beta : \mathbb{C}^n \rightarrow \mathbb{C}^m \). And since the image of \( \beta \) contains \( F \), for each \( \eta_j \), there is a unique \( \tilde{\eta}_j \in \mathbb{C}^m \), such that \( \beta(\tilde{\eta}_j) = \eta_j \). Let \( \tilde{\eta} = \sum_{j=1}^{n} \tilde{\eta}_j \otimes \varepsilon_j^{(n)} \). Then \( (\beta \otimes I_n)(\tilde{\eta}) = \sum_{j=1}^{n} \eta_j \otimes \varepsilon_j^{(n)} = \eta \).

\[\square\]

**Proposition 4.2.3.** If \( V \) is an abstract operator space and \( \varphi : V \rightarrow M_n \) is a linear mapping, then \( \|\varphi\|_{cb} = \|\varphi^{(n)}\| \).

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4.3. THE REPRESENTATION THEOREM

Proof. By Proposition 4.2.1, it suffices to show that \( \|\varphi^{(m)}\| \leq \|\varphi^{(n)}\| \) for any \( m > n \). Given \( \varepsilon > 0 \), choose \( v \in M_m(V) \) with \( \|v\| \leq 1 \) such that \( \|\varphi^{(m)}(v)\| - \varepsilon < \|\varphi^{(m)}(v)\| \). By definition, there exists a unit vector \( \eta \in (\mathbb{C}^n)^m \cong \mathbb{C}^m \otimes \mathbb{C}^n \) such that \( \|\varphi^{(m)}(v)\| = \|\varphi^{(m)}(v)(\eta)\| \). So, there is a unit vector \( \xi \in (\mathbb{C}^n)^m \) satisfying \( \|\varphi^{(m)}(v)(\eta)\| = |\langle \varphi^{(m)}(v)(\eta), \xi \rangle| \).

Thus we have \( \|\varphi^{(m)}(v)\| = |\langle \varphi^{(m)}(v)(\eta), \xi \rangle| \). By Lemma 4.2.2, there are isometries \( \alpha, \beta : \mathbb{C}^n \hookrightarrow \mathbb{C}^m \) and unit vectors \( \tilde{\xi}, \tilde{\eta} \in \mathbb{C}^m \otimes \mathbb{C}^n \) such that \( \xi = (\alpha \otimes I_n)(\tilde{\xi}) \) and \( \eta = (\beta \otimes I_n)(\tilde{\eta}) \). Then

\[
\|\varphi^{(m)}\| - \varepsilon < |\langle \varphi^{(m)}(v)(\eta), \xi \rangle| = |\langle \varphi^{(m)}(v)(\beta \otimes I_n)(\tilde{\eta}), (\alpha \otimes I_n)(\tilde{\xi}) \rangle| \\
= |\langle (\alpha \otimes I_n)^* \varphi^{(m)}(v)(\beta \otimes I_n)(\tilde{\eta}), (\tilde{\xi}) \rangle| \\
= |\langle (\alpha^* \varphi^{(m)}(v)(\beta), (\tilde{\xi}) \rangle| = |\langle \varphi^{(n)}(\alpha^* v \beta)(\tilde{\eta}), (\tilde{\xi}) \rangle| \\
\leq \|\varphi^{(n)}(\alpha^* v \beta)\| \leq \|\varphi^{(n)}\|
\]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that \( \|\varphi^{(m)}\| \leq \|\varphi^{(n)}\| \) for all \( m > n \). \( \Box \)

The following consequence of Proposition 4.2.3 is frequently used.

Corollary 4.2.4. If \( V \) is an abstract operator space, then for any linear functional \( f : V \rightarrow \mathbb{C} \), we have \( \|f\|_{\text{cb}} = \|f\| \).

4.3. The Representation Theorem

We have seen that any concrete operator space is an abstract operator space. More significantly, every abstract operator space is identified with a concrete operator. This theorem is due to Z.-J. Ruan, which is usually called Ruan's Representation Theorem.

Theorem 4.3.1. If \( V \) is an abstract operator space, then there is a Hilbert space \( H \), a concrete operator space \( W \subseteq B(H) \), and a complete isometry from \( V \) onto \( W \). If \( V \) is separable as a normed space, then we can let \( H = l_2 \).

Note. With this representation theorem, we can also identify an abstract operator space \( V \) with a subspace of \( B(H) \), and consequently,

\[
M_m(M_n(V)) \cong M_{m \times n}(V) \cong M_n(M_m(V))
\]

in the meaning of isometric isomorphism, where \( m, n \in \mathbb{N} \).

In order to prove Theorem 4.3.1, we need the following lemma for matrix-valued mapping which is analogous to Hahn-Banach theorem.
Lemma 4.3.2. Suppose that $V$ is an abstract operator space. Given any element $v \in M_n(V)$, there exists a complete contraction $\varphi : V \to M_n$, such that

$$
\|\varphi_n(v)\| = \|v\|
$$

With this lemma, we present the proof of the first part of Theorem 4.3.1. The rest of the proof can be found in [13, Theorem 2.3.5].

Proof. For each $n \in \mathbb{N}$, we let $s_n = s_n(V) = CB(V, M_n)_{\|\cdot\|_{\alpha \leq 1}}$ and $s(V) = \bigcup_{n \in \mathbb{N}} s_n(V)$. Each $S_n$ is not empty since $0 \in S_n$. Then we define $H = \bigoplus_{\varphi \in s} C^n(\varphi)$, where $n(\varphi)$ is the integer $n$ with $\varphi \in s_n$. Note that even if $\varphi_1 \neq \varphi_2$, $n(\varphi_1)$ may be equal to $n(\varphi_2)$. Let $\Phi : V \to B(H)$ by $\Phi(v) = (\varphi(v))_{\varphi \in s}$. For any $h \in H$, $h$ has the form $(h_\varphi)_{\varphi \in s}$. We have

$$
\Phi(v)(h) = (\varphi(v))_{\varphi \in s}(h) = (\varphi(v))_{\varphi \in s}(h_\varphi)_{\varphi \in s} = (\varphi(v)(h_\varphi))_{\varphi \in s},
$$

where $h_\varphi \in C^n(\varphi)$ and $\varphi : V \to M_n(V)$. Since $\Phi(v) = (\varphi(v))_{\varphi \in s}$, we can identify $\Phi_m : M_m(V) \to B(H^m)$ with the mapping $v \mapsto (\varphi_m(v))_{\varphi \in s}$. We also note that, for each $\varphi \in s$, $\varphi \in s_n$ for some $n \in \mathbb{N}$ and $\varphi$ is a mapping from $V$ to $M_n$. Thus $\varphi_m(v)$ is a mapping from $M_m(V)$ to $M_m(M_n)$ for each $m \in \mathbb{N}$. Since each $\varphi$ is completely contractive, it is immediate that $\Phi$ is a complete contraction with $\|\Phi_n(v)\| \leq \|v\|$, for all $n \in \mathbb{N}$.

On the other hand, given a fixed $v \in M_n(V)$, we may use Lemma 4.3.2 to select a $\varphi_0 \in s_n$ with $\|(\varphi_0)_n(v)\| = \|v\|$. This implies that $\|\Phi_n(v)\| \geq \|(\varphi_0)_n(v)\| = \|v\|$. Thus, $\|\Phi_n(v)\| = \|v\|$ for all $n \in \mathbb{N}$, and $\Phi$ is a complete isometry.

To show that $\varphi_n : M_n(B(H)) \to B(H^n)$ is surjective, define $\psi_i : H \to H^n$ by $\psi_i(h) = (0, \ldots, 0, \underbrace{h}_{i-th}, 0, \ldots, 0)$ for all $h \in H$ and $1 \leq i \leq n$. Then $\psi_i$ is an isomorphism from $H$ onto $H^i = \{(0, \ldots, 0, h, 0, \ldots, 0) : h \in H\}$ for each $1 \leq i \leq n$. For any $T \in B(H^n)$ and $h = (h_1, \ldots, h_n) \in H^n$, we have

$$
Th = T(\sum_{i=1}^{n} \hat{h}_i) = \sum_{i=1}^{n} T\hat{h}_i
$$

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with \( \tilde{h}_i = \psi_i(h_i) \). Note that \( T\tilde{h}_i \in H^n \) and \( T\tilde{h}_i = (T_1 \tilde{h}_i, ..., T_n \tilde{h}_i) = (b_{1i} h_i, ..., b_{ni} h_i) \), with \( b_{ji} = T_j \circ \psi_i \). So,

\[
T h = \sum_{i=1}^{n} T \tilde{h}_i = (\sum_{i=1}^{n} T_1 \tilde{h}_i, ..., \sum_{i=1}^{n} T_n \tilde{h}_i) = (\sum_{i=1}^{n} b_{1i} h_i, ..., \sum_{i=1}^{n} b_{ni} h_i) = \varphi_n(b)(h).
\]

\[\square\]

**Theorem 4.3.3.** Suppose that \( V \) is a linear space, and that we are provided with mappings

\[
\| \cdot \|_n : M_n(V) \rightarrow [0, \infty)
\]

for all \( n \in \mathbb{N} \), which satisfy

- \( M1' \) \( \| v \oplus w \|_{m+n} \leq \max\{\|v\|_m, \|w\|_n\} \),
- \( M2 \) \( \| \alpha v \beta \|_n \leq \|\alpha\| \|v\|_m \|\beta\| \),

for all \( v \in M_m(V), w \in M_n(V), \alpha \in M_{m,n} \) and \( \beta \in M_{n,n} \). Then these mappings are seminorms which satisfy \( M1 \) and \( M2 \). If, in addition, \( \| \cdot \|_1 \) is a norm, then the same is true for all the given matrix seminorms, and they determine an operator space structure on \( V \).

**Proof.** Given \( v, w \in M_n(V) \), and \( \varepsilon > 0 \), we let \( \alpha = \|v\|_n + \varepsilon \) and \( \beta = \|w\|_n + \varepsilon \).

Then \( v = \alpha \tilde{v} \) and \( w = \beta \tilde{w} \) for some \( \tilde{v}, \tilde{w} \in M_n(V) \) with \( \|\tilde{v}\|, \|\tilde{w}\| < 1 \). Then

\[
v + w = \gamma \begin{pmatrix} \tilde{v} & 0 \\ 0 & \tilde{w} \end{pmatrix} \gamma^*,
\]

where \( \gamma = [\alpha^{1/2} I_n, \beta^{1/2} I_n] \) and \( \gamma^* = \gamma^t \) is the adjoint of \( \gamma \) satisfying

\[
\|\gamma\| \|\gamma^*\| = \|\gamma\gamma^*\| = \alpha + \beta.
\]

We have

\[
\|v + w\| = \|\gamma \begin{pmatrix} \tilde{v} & 0 \\ 0 & \tilde{w} \end{pmatrix} \gamma^*\| \leq \|\gamma\| \|\begin{pmatrix} \tilde{v} & 0 \\ 0 & \tilde{w} \end{pmatrix}\| \|\gamma^*\|
\]

\[
= (\alpha + \beta) \|\begin{pmatrix} \tilde{v} & 0 \\ 0 & \tilde{w} \end{pmatrix}\| \leq \max\{\|\tilde{v}\|, \|\tilde{w}\|\} (\alpha + \beta)
\]

\[
< \alpha + \beta = \|v\| + \|w\| + 2\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, it follows that \( \|v + w\| \leq \|v\| + \|w\| \).
For any \( \alpha \in \mathbb{C} \), by the condition M2, we have on the one hand,
\[
\| \alpha v \| = \| \alpha I_n \circ v \circ I_n \| \leq \| \alpha I_n \| \| v \|_n \| I_n \| = |\alpha| \| v \|, 
\]
and on the other hand,
\[
\| v \|_n = \| -\alpha v \|_n \leq \| \alpha v \|. 
\]
Or equivalently, \( \| \alpha v \|_n \geq |\alpha| \| v \|_n \). Therefore, \( \| \alpha v \|_n = |\alpha| \| v \|_n \).

4.4. Some Usual Operator Spaces Induced by Given Operator Spaces

Suppose that \( V \) is an operator space. If \( V_I \) is a subspace of \( V \), then what is the induced operator space structure on \( V_I \)? In case that \( V_I \) is a closed subspace of \( V \), what is the natural operator space structure on the quotient space \( V/V_I \)? And if we are given operator spaces \( V \) and \( W \), what is the product operator space structure on \( V \times W \)? In this section, we will define a natural operator space structure on each of these spaces. In the sequel, when these spaces are considered as operator spaces, they are always equipped with the natural operator space structures defined in this section.

4.4.1. Subspaces. Let \( V \) be an operator space, \( W \subseteq V \) a subspace of \( V \). Then \( M_n(W) \) is a linear subspace of \( M_n(V) \) for all \( n \in \mathbb{N} \). It is easy to see that the corresponding norm on \( M_n(W) \) relative to \( M_n(V) \) determines a matrix norm on \( W \). Thus \( W \) is also an operator space as a subspace of \( V \). That is to say, any linear subspace of an operator space is also an operator space with the induced matrix norms. To see that, let \( \| \cdot \|_n \) denote the norm on \( M_n(V) \) and \( \| \cdot \|_n^w \) denote the norm on \( M_n(W) \) restricted from \( M_n(V) \). Then for \( a \in M_n(W) \) and \( b \in M_n(W) \), we have \( a \oplus b \in M_{n+m}(W) \subseteq M_{n+m}(V) \). So,
\[
\| a \oplus b \|_{n+m}^w = \| a \oplus b \|_{n+m} \leq \max\{\| a \|_m, \| b \|_n \} = \max\{\| a \|_m^w, \| b \|_n^w \}. 
\]
This proved M1. For \( v \in M_m(W) \), \( \alpha \in M_{m,m} \) and \( \beta \in M_{m,n} \), we have
\[
\| \alpha v \beta \|_n^w = \| \alpha v \beta \|_n \leq \| \alpha \|_m \| v \|_m \| \beta \|_n = \| \alpha \| \| v \|_m^w \| \beta \|_n. 
\]
This proved M2. So, the matrix norm sequence \( \{\| \cdot \|_n^w : n \in \mathbb{N} \} \) is an operator space norm, and \( W \) is an operator space with this norm, called an operator subspace of \( V \).
4.4. SOME USUAL OPERATOR SPACES INDUCED BY GIVEN OPERATOR SPACES

4.4.2. Matrix Spaces. If \( V \) is an operator space, then for any \( p \in \mathbb{N} \), we define a matrix norm on \( M_p(V) \) by the identification:

\[
M_m(M_p(V)) \cong M_{mp}(V) : (v_{i,j})_{k,h} \mapsto \tilde{v}_{(i-1)p+k,(j-1)p+h}
\]

for \( m, i, j, k, h \in \mathbb{N} \).

For any \( \tilde{v}_m^p \in M_m(M_p(V)) \) and \( \tilde{v}_n^p \in M_n(M_p(V)) \), we have

\[
\| \tilde{v}_m^p \oplus \tilde{v}_n^p \|_{m+n} = \| \tilde{v}_m^p \oplus \tilde{v}_n^p \|_{(m+n)p} = \max\{\| \tilde{v}_m^p \|_m, \| \tilde{v}_n^p \|_n\}
\]

This proved M1. On the other hand, let \( \alpha \in M_{m,m}, \beta \in M_{m,n} \), and \( \nu \in M_m(M_p(V)) \). With the identification \( M_m(M_p(V)) \cong M_{mp}(V) \), we have

\[
\| \alpha \nu \beta \| = \| (\alpha \otimes I_p) \nu (\beta \otimes I_p) \| \leq \| \alpha \otimes I_p \| \| \nu \| \| \beta \otimes I_p \| = \| \alpha \| \| \nu \| \| \beta \|.
\]

This proved M2. Thus \( M_p(V) \) is also an operator space with the induced matrix norm.

4.4.3. Quotient Spaces. Suppose \( V \) is an operator space, and \( N \subseteq V \) is a closed subspace. Then \( M_n(N) \) is closed in \( M_n(V) \).

If we define a mapping \( \| \cdot \|_q : M_n(V)/M_n(N) \rightarrow \mathbb{R} \) by

\[
\| \tilde{v} \|_q = \inf\{\| v + w \| : w \in M_n(N)\} = \inf\{\| v \| : v \in M_n(V), \pi_n(v) = \tilde{v}\},
\]

then \( \| \cdot \|_q \) is a norm on \( M_n(V)/M_n(N) \). If we only regard \( M_n(V/N) \) and \( M_n(V)/M_n(N) \) as two linear spaces, then the mapping \( (v_{i,j})_{n \times n} \mapsto (v_{i,j})_{n \times n} + M_n(N) \) is an isomorphism, i.e. \( M_n(V/N) \cong M_n(V)/M_n(N) \). With this identification, we can induce a norm on \( M_n(V/N) \) from \( M_n(V)/M_n(N) \). We prove that the matrix norms on \( M_n(V/N), (n \in \mathbb{N}) \) in this way satisfy M1 and M2.

Given \( \alpha \in M_{m,m}, \beta \in M_{m,n} \), and \( \tilde{v} \in M_m(V/N) \), there is a \( v \in M_m(V) \) such that \( \pi_m(v) = \tilde{v} \) and \( \| v \|_m < \| \tilde{v} \|_m^q + \varepsilon \). Also note that \( \pi_n(\alpha \nu \beta) = \alpha \tilde{v} \beta \), and thus

\[
\| \alpha \tilde{v} \beta \|_n^q \leq \| \alpha \nu \beta \|_n \leq \| \alpha \| \| v \|_m \| \beta \| \leq \| \alpha \| (\| \tilde{v} \|_m^n + \varepsilon) \| \beta \|.
\]

Since \( \varepsilon \) is arbitrary, M2 holds.

On the other hand, given \( \tilde{v}, \tilde{w} \in M_n(V/N) \) there are \( v, w \in M_n(V) \) such that \( \pi_n(w) = \tilde{w}, \pi_n(v) = \tilde{v} \) \| v \| \leq \| \tilde{v} \|_n^q + \varepsilon \) and \( \| w \|_n \leq \| \tilde{w} \|_n^q + \varepsilon \). Also notice that
4.4. SOME USUAL OPERATOR SPACES INDUCED BY GIVEN OPERATOR SPACES

\( \pi_{m+n}(v \oplus w) = \pi_m(v) \oplus \pi_n(w) = \tilde{v} \oplus \tilde{w}, \) thus we have

\[ \| \tilde{v} \oplus \tilde{w} \| \leq \| v \oplus w \| = \max\{ \| v \|, \| w \| \} \leq \max\{ \| \tilde{v} \|, \| \tilde{w} \| \} + \varepsilon. \]

Since \( \varepsilon \) is arbitrary, we get \( M1' \).

Thus, the quotient space \( V/N \) with the matrix norms defined in this way is also an operator space, called the quotient operator space determined by \( V \) and \( W \). In this case, the quotient mapping \( \pi : V \rightarrow V/N \) is a complete quotient mapping since, given \( v \in M_n(V), \varphi_n(v) = (\varphi(v_{ij}))_{n \times n} \in M_n(V/N), \) and under the identification \( M_n(V/N) \cong M_n(V)/M_n(N), \varphi_n(v) \) is a quotient mapping for all \( n \in \mathbb{N} \).

4.4.4. Product of Operator Spaces. Given an indexed family of operator spaces \( (V_s)_{s \in S} \), we define the product operator space \( \prod_{s \in S} V_s \) to be the normed space \( l_\infty(S; V_s) \) together with the matrix norms determined by the identifications

\[ M_n(\prod_{s \in S} V_s) = \prod_{s \in S} M_n(V_s) = l_\infty(S; M_n(V_s)). \]

We will prove that, \( V = \prod_{s \in S} V_s \) with the defined matrix norms is really an operator space.

Recall that the norm of \( x \in l_\infty(S; V_s) \) is defined by \( \| x \| = \sup \{ \| x_s \| : s \in S \} \), where \( x = (x_s)_{s \in S} \). Let \( m, n \in \mathbb{N}, v \in M_n(V) \) and \( w \in M_m(V) \). Then we have

\[ \| u \oplus w \|_{m+n} = \sup \{ \| (u \oplus w)_s \|_{m+n} : s \in S \} = \sup \{ \max \{ \| u_s \|_n, \| w_s \|_m \} : s \in S \} \]
\[ = \max \{ \sup \{ \| u_s \|_n : s \in S \}, \sup \{ \| w_s \|_m : s \in S \} \} \]
\[ = \max \{ \| u \|_n, \| w \|_m \}. \]

This proved \( M1 \). Let \( \alpha \in M_{m,n}, \beta \in M_{n,m}. \) We have

\[ \alpha v \beta = \alpha(v_{ij}) \beta = \alpha(\prod_{s \in S} v_{ij}) \beta \cong \alpha(\prod_{s \in S} v_s) \beta = \prod_{s \in S} \alpha v_s \beta, \]

where \( v_s \in M_n(V_s) \). So,

\[ \| \alpha \| \sup \{ \| u_s \| : s \in S \} \| \beta \| \leq \sup \{ \| \alpha \| \| u_s \| \| \beta \| : s \in S \} \]
\[ = \| \alpha \| \sup \{ \| u_s \| : s \in S \} \| \beta \| = \| \alpha \| \| v \| \| \beta \|. \]
This proved M2. Thus, \( \prod_{s \in S} V_s \) with the matrix norms determined by the identifications

\[
M_n\left( \prod_{s \in S} V_s \right) = \prod_{s \in S} M_n(V_s) = l_\infty(S; M_n(V_s)) \ n \in \mathbb{N}
\]

is an operator space, called the \textit{product operator space} of \( V_s, \ s \in S \).

4.5. Dual Space and Mapping Spaces

The purpose of this section is to find a natural operator space structure on the mapping space \( CB(V, W) \) for operator spaces \( V \) and \( W \). For simplicity, we first consider the case \( W = \mathbb{C} \), i.e., \( V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C}) \).

\textbf{Theorem 4.5.1.} If \( V \) is an operator space, then \( V^* \) is an operator space via the identifications \( M_n(V^*) \cong CB(V, M_n), \ n \in \mathbb{N} \).

\textbf{Proof.} We first show the linear space identification \( M_n(V^*) \cong CB(V, M_n), \ n \in \mathbb{N} \). Then we use this identification to define a matrix norm on \( M_n(V^*) \) \( (n \in \mathbb{N}) \). Finally, we show that \( V^* \) with this matrix norm is an operator space. For any \( f = (f_{i,j}) \in M_n(V^*) \), where \( f_{i,j} \in V^* = CB(V, \mathbb{C}) \), it determines a linear mapping \( \tilde{f} : V \rightarrow M_n \) by \( \tilde{f}(v) = [f_{ij}(v)] \). We want to show that the map \( \psi : f \rightarrow \tilde{f} \) is a linear isomorphism from \( M_n(V^*) \) onto \( CB(V, M_n) \). By Proposition 4.2.3, \( \tilde{f} \in CB(V, M_n) \) and \( \|\tilde{f}\|_{cb} = \|\tilde{f}\|_n \). So \( \psi(M_n(V^*)) \subseteq CB(V, M_n) \). It is easy to see that this map is one-one. On the other hand, for any \( T \in CB(V, M_n) \), we have \( T(v) = [a_{ij}(v)]_n \) for all \( v \in V \). Then each \( a_{ij} \) is a linear mapping from \( V \) to \( \mathbb{C} \). Moreover, for \( 1 \leq i_0, j_0 \leq n \),

\[
\|a_{i_0, j_0}(v)\| \leq \|a_{ij}(v)\| = \|T(v)\|
\]

for all \( v \in V \). This shows that each \( a_{ij} \in V^* \). Let \( a = [a_{ij}] \in M_n(V^*) \). Since \( \psi(a) = T \), \( \psi \) is onto.

We use \( CB(V, M_n) \) to define a norm on \( M_n(V^*) \) for each \( n \in \mathbb{N} \). In the following, we prove that \( V^* \) is an operator space with this matrix norm. Let \( f \in M_m(V^*), \alpha \in M_{n,m} \) and \( \beta \in M_{m,n} \). We want to prove \( \|\alpha f \beta\|_{M_n(V^*)} \leq \|\alpha\| \|f\|_{M_m(V^*)} \|\beta\|. \) Since \( f \in M_m(V^*) \cong CB(V, M_m) \), by definition, \( ||f||_{M_m(V^*)} = ||f||_{CB(V,M_m)} \). Similarly, \( ||\alpha f \beta||_{M_n(V^*)} = ||\alpha f \beta||_{CB(V,M_n)} \). So, it suffices to show that

\[
||\alpha f \beta||_{cb} \leq ||\alpha|| \|f\|_{cb} \|\beta||.
\]
For any $r \in \mathbb{N}$ and $v \in M_r(V)$, we have

$$ (\alpha f(\beta)^{(r)}(v) = \begin{pmatrix} \alpha f(\beta(v_{n1}) & \cdots & \alpha f(\beta(v_{nr})) \\ \vdots & \ddots & \vdots \\ \alpha f(\beta(v_{r1}) & \cdots & \alpha f(\beta(v_{rr})) \end{pmatrix}_{r \times r} = \begin{pmatrix} \alpha f(\beta(v_{n1}) \beta & \cdots & \alpha f(\beta(v_{nr}) \beta) \\ \vdots & \ddots & \vdots \\ \alpha f(\beta(v_{r1}) \beta & \cdots & \alpha f(\beta(v_{rr}) \beta) \end{pmatrix}_{r \times r} $$

$$ = \begin{pmatrix} \alpha & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha \end{pmatrix} \begin{pmatrix} f(v_{n1}) & \cdots & f(v_{nr}) \\ \vdots & \ddots & \vdots \\ f(v_{r1}) & \cdots & f(v_{rr}) \end{pmatrix} \begin{pmatrix} \beta & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta \end{pmatrix} $$

$$ = (\alpha \otimes I^{(r)}(f^{(r)}(v))(\beta \otimes I^{(r)}). $$

This is a composition of linear operators over $\mathbb{C}^n$. Thus,

$$ \| (\alpha f(\beta)^{(r)}(v) \| \leq \| \alpha \otimes I^{(r)} \| \| f^{(r)}(v) \| \| \beta \otimes I^{(r)} \| $$

$$ = \| \alpha \| \| f^{(r)}(v) \| \| \beta \| \leq \| \alpha \| \| f^{(r)} \| \| v \| \| \beta \| $$

$$ \leq \| \alpha \| \| f \|_{ch} \| \beta \| \| v \|. $$

Therefore, $\| (\alpha f(\beta)^{(r)} \| \leq \| \alpha \| \| f \|_{ch} \| \beta \|$ for all $r \in \mathbb{N}$. So

$$ \| \alpha f(\beta) \|_{ch} \leq \| \alpha \| \| f \|_{ch} \| \beta \|. $$

This proved M2.

On the other hand, given $f \in M_m(V^*)$, $g \in M_n(V^*)$, and $v \in M_r(V)$ with $\|v\| \leq 1$, we have

$$ \| (f \oplus g)^{(r)}(v) \| = \left\| \begin{pmatrix} f \oplus g(v_{n1}) & \cdots & f \oplus g(v_{nr}) \\ \vdots & \ddots & \vdots \\ f \oplus g(v_{r1}) & \cdots & f \oplus g(v_{rr}) \end{pmatrix} \right\| $$

$$ = \left\| \begin{pmatrix} f(v_{n1}) \oplus g(v_{n1}) & \cdots & f(v_{nr}) \oplus g(v_{nr}) \\ \vdots & \ddots & \vdots \\ f(v_{r1}) \oplus g(v_{r1}) & \cdots & f(v_{rr}) \oplus g(v_{rr}) \end{pmatrix} \right\| $$

$$ = \left\| \begin{pmatrix} f(v_{n1}) & \cdots & f(v_{nr}) \\ \vdots & \ddots & \vdots \\ f(v_{r1}) & \cdots & f(v_{rr}) \end{pmatrix} \oplus \begin{pmatrix} g(v_{n1}) & \cdots & g(v_{nr}) \\ \vdots & \ddots & \vdots \\ g(v_{r1}) & \cdots & g(v_{rr}) \end{pmatrix} \right\| $$

$$ = \| f^{(r)}(v) \| \oplus g^{(r)}(v) \|. $$
Note that, \( f^{(r)}(v) \in M_r(M_m) \cong M_{rm} \), similarly, \( g^{(r)}(v) \in M_r \), so
\[
\| f^{(r)}(v) + g^{(r)}(v) \| = \max\{\| f^{(r)}(v) \|, \| g^{(r)}(v) \|\} \leq \max\{\| f^{(r)} \|, \| g^{(r)} \|\}
\]
\[
\leq \max\{\| f \|_{cb}, \| g \|_{cb}\}.
\]
That is, for any \( r \in \mathbb{N} \), \( \| (f + g)^{(r)} \| \leq \max\{\| f \|_{cb}, \| g \|_{cb}\} \), and so
\[
\| f \oplus g \|_{cb} \leq \max\{\| f \|_{cb}, \| g \|_{cb}\}
\]
This proved \( M1' \). Therefore, \( V^* \) is an operator space via the linear space identifications \( M_n(V^*) \cong CB(V, M_n) \) \( n \in \mathbb{N} \).

With this operator space structure, \( V^* \) is called the dual operator space of \( V \).

With the above argument, we see that \( M_n(V^*) \cong CB(V, M_n) \) is also an operator space identification for each \( n \in \mathbb{N} \).

**Theorem 4.5.2.** If \( V \) and \( W \) are operator spaces, then \( CB(V, W) \) is an operator space via the linear space identifications \( M_m(CB(V, W)) \cong CB(V, M_m(W)) (m \in \mathbb{N}) \).

**Proof.** We first prove that \( M_m(CB(V, W)) \) is linear isomorphic to \( CB(V, M_m(W)) \). So we can define norms on \( M_m(CB(V, W)) \) using the norms on \( CB(V, M_m(W)) \) for all \( m \in \mathbb{N} \).

Define \( \psi : M_n(CB(V, W)) \to L(V, M_n(W)) \) by
\[
\psi(\varphi)(v) = [\varphi_{ij}(v)]_{n \times n}
\]
for all \( v \in V \) and \( \varphi \in M_n(CB(V, W)) \). We will show that \( \psi \) is a linear isomorphism from \( M_n(CB(V, W)) \) onto \( CB(V, M_n(W)) \). It is easy to see that \( \psi \) is linear and injective. It suffices to show that \( \psi(\varphi) \in CB(V, M_n(W)) \) for all \( \varphi \in M_n(CB(V, W)) \), and \( \psi(M_n(CB(V, W))) = CB(V, M_n(W)) \).

Let \( \varphi = [\varphi_{ij}]_{n \times n} \in M_n(CB(V, W)) \). Each \( \varphi_{ij} \) satisfies \( \| (\varphi_{ij})^{(r)} \| \leq \| \varphi_{ij} \|_{cb} \) for all \( r \in \mathbb{N} \) and \( 1 \leq i, j \leq n \). For any \( r \in \mathbb{N} \) and \( v \in M_r(V) \),
\[
(\psi(\varphi))^{(r)}(v) = (\psi(\varphi))^{(r)}([v_{ij}]) = [\psi(\varphi)(v_{ij})]_{r \times r} = ((\varphi_{hk}(v_{ij}))_{n \times n})_{r \times r}
\]
\[
\cong ((\varphi_{hk}(v_{ij}))_{r \times r})_{n \times n} = [((\varphi_{hk})^{(r)}(v)]_{n \times n},
\]

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where we write $A \cong B$ if there are unitary matrices $U_1, U_2$ such that $B = U_1 A U_2$. So,

$$
\|(\psi(\varphi))^{(r)}(v)\| = \|((\varphi_{hk})^{(r)}(v))_{n \times n}\| \leq \sum_{h,k=1}^{n} \|(\varphi_{hk})^{(r)}(v)\| \\
\leq \sum_{h,k=1}^{n} \|(\varphi_{hk})^{(r)}\|\|v\| \leq \sum_{h,k=1}^{n} c_{hk} \|v\| = (\sum_{h,k=1}^{n} c_{hk})\|v\| = c\|v\|,
$$

where $c_{hk} = \|\varphi_{hk}\|_{cb}$. Since $c$ does not depend on $r$, $\psi(\varphi)$ is completely bounded. This proved that $\psi(\varphi) \in CB(V, M_n(V, W))$ for all $\varphi \in M_n(CB(V, W))$.

On the other hand, let $T \in CB(V, M_n(W))$, and $v \in V$. Then $T(v) = [a_{ij}(v)]$, where each $a_{ij}(v) \in W$. It is easy to see that each $a_{ij}$ is linear on $V$. Now we show that each $a_{ij}$ is also completely bounded, and so $a = [a_{ij}]_{n \times n} \in M_n(CB(V, W))$. Since $\psi(a) = T$, $\psi$ is onto. For any $r \in N$, and $v \in M_r(V)$,

$$
T^{(r)}(v) = [T(v_{hk})]_{r \times r} = [(a_{ij})_{n \times n}(v_{hk})]_{r \times r} = ((a_{ij}(v_{hk}))_{n \times n})_{r \times r}
$$

$$
= [(a_{ij}(v))_{r \times r}]_{n \times n} = [(a_{ij})^{(r)}(v)]_{n \times n}.
$$

Thus, for all $1 \leq i_0, j_0 \leq n$, we have

$$
\|(a_{i_0 j_0})^{(r)}(v)\| \leq \|([a_{ij}]^{(r)}(v))\| = \|T^{(r)}(v)\| \leq \|T^{(r)}\|\|v\| \leq \|T\|_{cb}\|v\|.
$$

So $\|(a_{ij})^{(r)}\| \leq \|T\|_{cb}$. Since $r$ is arbitrary and $\|T\|_{cb}$ does not depend on $r$, each $a_{ij}$ must be completely bounded. This shows $\psi$ is onto. So, $M_m(CB(V, W)) \cong CB(V, M_m(W))$. By the same arguments as used in the case of $V^*$, it can be shown that $CB(V, W)$ has an operator space structure via this identification. □

It is well-known that, for a normed space $V$, the canonical inclusion $\iota_V : V \hookrightarrow V^{**}$ defined by

$$
\langle \iota_V(v), f \rangle = \langle f, v \rangle
$$

is an isometry. In the case of operator space we have the similar result.

Let $V$ be an operator space, and $V^*$ the dual operator space of $V$. Then the matrix pairing $\langle \cdot, \cdot \rangle : M_{m,n}(V) \times M_{p,q}(V^*) \rightarrow M_{mp,nq}$ is defined by

$$
\langle \langle v, w \rangle \rangle = [\langle v_{ij}, w_{hk} \rangle]_{mp,nq}
$$

for all $v \in M_{m,n}(V)$ and $w \in M_{p,q}(V^*)$.\n
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Theorem 4.5.3. For any operator space $V$, the canonical inclusion $i_V : V \hookrightarrow V^{**}$ is completely isometric.

Before proving this proposition, we emphasize that, although $M_n(V^*) \neq (M_n(V))^*$, the norm on $M_n(V)$ does determine the norm on $M_n(V^*)$, and vice versa. For any $f \in M_n(V^*)$, we have from Proposition 4.2.3 that

$$
\|f\|_{cb} = \|f^{(n)}\| = \sup\{\|f^{(n)}(v)\| : v \in M_n(V), \|v\| \leq 1\} \quad (22)
$$

$$
= \sup\{\|\langle f, v \rangle\| : v \in M_n(V), \|v\| \leq 1\}. \quad (23)
$$

On the other hand, for any $v \in M_n(V)$, and any $f \in CB(V, M_n) \cong M_n(V^*)$ with $\|f\|_{cb} \leq 1$,

$$
\|f^{(n)}(v)\| \leq \|f^{(n)}\| \|v\| \leq \|f\|_{cb} \|v\| \leq \|v\|.
$$

By Lemma 4.3.2, there exists a particular $f \in CB(V, M_n)$ with $\|f\| = 1$ such that $\|f^{(n)}(v)\| = \|v\|$. So,  

$$
\|v\| = \sup\{\|f^{(n)}(v)\| : f \in CB(V, M_n), \|f\|_{cb} \leq 1\} \quad (24)
$$

$$
= \sup\{\|\langle f, v \rangle\| : v \in M_n(V^*), \|f\| \leq 1\}. \quad (25)
$$

Proof. Now, we prove Proposition 4.5.3. For any $n \in \mathbb{N}$ and $v \in M_n(V)$, we want to prove $\|(i_V)^{(n)}(v)\| = \|v\|$. Note that

$$(i_V)^{(n)}v = [i_V(v_{ij})]_{n \times n} \in M_n(V^{**}) \cong CB(V^*, M_n).
$$

So, $\|(i_V)^{(n)}v\|_{M_n(V^{**})} = \|(i_V)^{(n)}v\|_{CB(V^*, M_n)} := \|(i_V)^{(n)}v\|_{cb}$.

By Proposition 4.2.3, $\|(i_V)^{(n)}v\|_{cb} = \|((i_V)^{(n)}(v))^{(n)}\|$. For any $f \in M_n(V^*)$, we have

$$(i_V)^{(n)}(v)^{(n)}(f) = [[(i_V)^{(n)}(v)(f_{jk})]_{n \times n} = [[(i_V)(v_{ij})]_{n \times n}(f_{jk})]_{n \times n}
$$

$$
= [f_{jk}(v_{ij})]_{n \times n} = \langle f, v \rangle.
$$

By Equality (25), it follows that

$$
\|(i_V)^{(n)}(v)\|_{cb} = \|(i_V)^{(n)}(v))_{cb} = \sup\{\|\langle f, v \rangle\| : f \in M_n(V^*), \|f\| \leq 1\} = \|v\|.
$$

Thus, $(i_V)^{(n)}v$ is isometric for each $n$, and $i_V$ is a completely isometry.

Proposition 4.5.4. Given operator spaces $V$ and $W$, and a completely bounded mapping $\varphi : V \longrightarrow W$, we have $\|(\varphi^*)^{(n)}\| = \|\varphi^{(n)}\|$ for all $n \in \mathbb{N}$.
4.5. DUAL SPACE AND MAPPING SPACES

PROOF. By definition, we have

\[ \|(\varphi^*)^{(n)}\| = \sup\{\|(\varphi^*)^{(n)}(g)\| : g \in M_n(W^*), \|g\| \leq 1\}. \]

Note that \((\varphi^*)^{(n)}(g) \in M_n(V^*)\), so by Equality (23),

\[ \|(\varphi^*)^{(n)}\| = \sup\{\|(\varphi^*)^{(n)}(g, v)\| : v \in M_n(V), \|v\| \leq 1 : g \in M_n(V^*), \|g\| \leq 1\} \]

Note that,

\[ \|(\varphi^*)^{(n)}(g, v)\| = \|[\varphi^*(g)](v)\| = \|[\varphi^*(g)]_{n^2} = g_{ij}(v_{jk})\|_{n^2} \cong \|(g, \varphi^{(n)}(v))\|. \]

So, \(\|(\varphi^*)^{(n)}(g, v)\| = \|(g, \varphi^{(n)}(v))\|\). Then

\[ \|(\varphi^*)^{(n)}\| = \sup\{\|(\varphi^*(g, v))\| : g \in M_n(V^*), v \in M_n(V), \|g\| \leq 1, \|v\| \leq 1\} \]

By Equality (25), we have

\[ \|(\varphi^*)^{(n)}\| = \sup\{\|\varphi^{(n)}(v)\| : v \in M_n(V), \|v\| \leq 1\} = \|\varphi^{(n)}\|. \]

\[ \Box \]

Proposition 4.5.5. Let \(V\) and \(W\) be two operator spaces. If \(W\) is complete, then so is \(CB(V, W)\).

Let us suppose that \(W\) is complete. It suffices to show that \(CB(V, W)\) is a closed subspace of \(B(V, W)\) since the later one is complete. Given any Cauchy sequence \(\{\varphi_n\} \subseteq CB(V, W)\), it is clear that \(\{\varphi_n\}\) is a Cauchy sequence of bounded linear operators in space \(B(V, W)\). Since \(B(V, W)\) is complete, there exists a bounded linear mapping \(\varphi : V \to W\) such that \(\varphi_n\) converges to \(\varphi\) in the norm topology, i.e.

\[ \|\varphi_n - \varphi\|_{cb} \to 0. \]

Since \(\{\varphi_n\}\) is Cauchy in \(CB(V, W)\), for any \(\varepsilon > 0\), there exists a sufficiently large integer \(N(\varepsilon) > 0\) such that whenever \(n, m > N(\varepsilon)\), we have

\[ \|\varphi_n - \varphi_m\|_{cb} < \varepsilon. \]
Given any $v = [v_{ij}] \in M_p(V)$ and $p \in \mathbb{N}$, we have

$$
\|\varphi_n - \varphi_m\|_{CH(V)} \leq \varphi_n - \varphi_m\|_{CH(V)} < \varepsilon \|v\|.
$$

Since $W$ is complete, $\varphi_m(v_{ij})$ converges to $\varphi(v_{ij})$ in $W$. Then

$$
\|\varphi_n - \varphi_m\|_{CH(V)} \leq \varepsilon \|v\|,
$$

and thus $\|\varphi_n - \varphi\| \leq \varepsilon$. It follows that $\varphi \in CB(V, W)$ and $\varphi_n$ converges to $\varphi$ in $CB(V, W)$. So, $CB(V, W)$ is complete.
CHAPTER 5

Projective Tensor Products

In this chapter, after a review of projective tensor products of Banach spaces, the projective tensor product of operator spaces are discussed. We cite most of the materials from the book [13]. We try to find some similarities for projective tensor products between Banach space case and operator space case.

If $X$ and $Y$ are normed spaces, then we have different ways to define a norm on the algebraic tensor product $X \otimes Y$. The classical tensor product norms are the projective tensor product norm and the injective tensor product norm. Correspondingly, we can define the similar tensor product norms for operator spaces. In this section, we first recall Banach space projective tensor products. Then we introduce the operator space projective tensor product norm and prove that it is the largest operator space subcross matrix norm. After that, we introduce the joint amplifications of a bilinear map and its joint completely bounded norm. We show that there is a completely isometrical identification between the operator space projective tensor product and the space of jointly completely bounded bilinear maps. We also show the "projectivity" of operator space projective tensor products.

We also prove that for any Hilbert spaces $H$ and $K$, we have a natural complete isometry $B(H)_* \hat{\otimes} B(K)_* \cong B(H \hat{\otimes} K)_*$. Finally, we generalize this identification to the case of von Neumann algebras.

5.1. Projective Tensor Products of Banach Spaces

In this section, the definition of projective tensor product of Banach spaces is introduced first. Then some of its most fundamental properties are presented.

**Definition 5.1.1.** Given Banach spaces $E$ and $F$, a norm $\| \cdot \|_\mu$ on $E \widehat{\otimes} F$ is said to be a subcross norm (resp. cross norm) if $\| x \otimes y \|_\mu \leq \| x \| \| y \|$ (resp. $\| x \otimes y \|_\mu = \| x \| \| y \|$) for all $x \in E$ and $y \in F$. 

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5.1. PROJECTIVE TENSOR PRODUCTS OF BANACH SPACES

Theorem 5.1.2. Given Banach spaces $E$ and $F$ and $u \in E \otimes F$, let

$$
\|u\|_7 = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i, x_i \in E, y_i \in F, n \in \mathbb{N} \right\}.
$$

Then $\| \cdot \|_7$ is the largest subcross norm on $E \otimes F$, called the projective tensor product norm on $E \otimes F$. Moreover, $\| \cdot \|_7$ is a cross norm on $E \otimes F$.

Proof. It is easy to show that $\| \cdot \|_7$ is a seminorm. The trick to see that it is also a norm can be found in the proof for the case of operator space projective tensor product. So we omit this part of proof here.

(1) By definition, it is clear that $\|x \otimes y\|_7 \leq \|x\| \|y\|$. So $\| \cdot \|_7$ is a subcross norm.

Let $\| \cdot \|_\mu$ be any subcross norm on $E \otimes F$. For $u = \sum x_i \otimes y_i \in E \otimes F$, since $\| \cdot \|_\mu$ is a subcross norm, we have $\|u\|_\mu = \| \sum x_i \otimes y_i\|_\mu \leq \sum \|x_i \otimes y_i\|_\mu \leq \sum \|x_i\| \|y_i\|$. This shows that $\|u\|_\mu \leq \|u\|_7$, i.e., $\| \cdot \|_7$ is the largest subcross norm on $E \otimes F$.

(2) Let $x \in E$ and $y \in F$. Denote by $B_{E^*}$ and $B_{F^*}$ the unit balls of $E^*$ and $F^*$, respectively. Choose $\varphi \in B_{E^*}$ and $\psi \in B_{F^*}$ such that $\varphi(x) = \|x\|$ and $\psi(y) = \|y\|$. Consider the bounded bilinear form $J$ on $E \times F$ given by $J(x', y') = \varphi(x')\psi(y')$. Its linearization $\tilde{J}$ is a linear functional on $E \otimes F$. For any $u = \sum x_i \otimes y_i \in E \otimes F$,

$$
|\tilde{J}(u)| = |\tilde{J}(\sum_{i=1}^{n} x_i \otimes y_i)| \leq \sum_{i=1}^{n} |\tilde{J}(x_i \otimes y_i)| = \sum_{i=1}^{n} |\varphi(x_i)\psi(y_i)| \leq \sum_{i=1}^{n} \|x_i\| \|y_i\|,
$$

which implies that $|\tilde{J}(u)| \leq \|u\|_7$, for all $u \in E \otimes F$. Therefore, $\tilde{J}$ is a bounded linear functional on the normed space $(E \otimes F, \| \cdot \|_7)$ with norm at most one. Hence $\|x\| \|y\| = \tilde{J}(x \otimes y) \leq \|x \otimes y\|_7$. Combining this inequality with (1), we have $\|x \otimes y\|_7 = \|x\| \|y\|$. Therefore, $\| \cdot \|_7$ is a cross norm on $E \otimes F$. □

Let $E, F$ and $G$ be Banach spaces. We use $E \otimes_\gamma F$ to denote the normed space $(E \otimes F, \| \cdot \|_7)$, and use $E \otimes^\gamma F$ to denote the completion of $E \otimes_\gamma F$, called the projective tensor product of $E$ and $F$. If $\varphi : E \times F \to G$ is a bilinear mapping, then we define

$$
\| \varphi \| = \sup \{ \| \varphi(x, y) \| : x \in E, y \in F, \|x\|, \|y\| \leq 1 \}.
$$

(26)

We let $BBL(E \times F, G)$ denote the linear space of all such mappings $\varphi$ with $\| \varphi \| < \infty$. Clearly, $BBL(E \times F, G)$ is a normed space.
Theorem 5.1.3. Let $E, F$ and $G$ be Banach spaces. Then we have the following isometric identifications.

\[ B(E \otimes \gamma, F, G) \cong BBL(E \times F, G) \cong B(E, B(F, G)). \]  

(27)

Proof. Note that we have the following linear space identifications:

\[ L(E \otimes F, G) \cong BL(E \times F, G) \cong L(E, L(F, G)). \]  

(28)

(1) The linear isomorphism $\Psi : L(E \times F, G) \to L(E \otimes F, G)$ is given by

\[ \Psi(\varphi)(x \otimes y) = \varphi(x, y) \]

for all $\varphi \in L(E \times F, G), x \in E$ and $y \in F$. So, for any $\varphi \in BBL(E \times F, G)$, we have

\[ \|\varphi\| = \sup\{\|\varphi(x, y)\| : x \in E, y \in F, \|x\| \leq 1, \|y\| \leq 1\} = \sup\{\|\Psi(\varphi)(x \otimes y)\| : x \in E, y \in F, \|x\| \leq 1, \|y\| \leq 1\} \leq \sup\{\|\Psi(\varphi)(x \otimes y)\| : \|x \otimes y\|_{\gamma} \leq 1, x \in E, y \in F\} \leq \|\Psi(\varphi)\|, \]

since $\|\cdot\|_{\gamma}$ is a cross norm. Therefore, $\|\varphi\| \leq \|\Psi(\varphi)\| < \infty$.

On the other hand, for $u = \sum_{i=1}^{n} x_{i} \otimes y_{i} \in E \otimes \gamma F$,

\[ \Psi(\varphi)(u) = \sum_{i=1}^{n} \Psi(\varphi)(x_{i} \otimes y_{i}) = \sum_{i=1}^{n} \varphi(x_{i}, y_{i}). \]

Then

\[ \|\Psi(\varphi)(u)\| = \|\sum_{i=1}^{n} \varphi(x_{i}, y_{i})\| \leq \sum_{i=1}^{n} \|\varphi\| \|x_{i}\| \|y_{i}\| = (\sum_{i=1}^{n} \|x_{i}\| \|y_{i}\|) \|\varphi\|. \]

Thus $\|\Psi(\varphi)(u)\| \leq \|\varphi\| \|u\|_{\gamma}$. Since $u$ is arbitrary, we have $\|\Psi(\varphi)\| \leq \|\varphi\|$. Therefore, $\Psi$ maps $BBL(E \times F, G)$ isometrically onto $B(E \otimes \gamma F, G)$.

(2) The linear isomorphism $\Phi : L(E \times F, G) \to L(E, L(F, G))$ is defined by $\Phi(\varphi)(x)(y) = \varphi(x, y)$. So

\[ \|\Phi(\varphi)\| = \sup\{\|\Phi(\varphi)(x)\| : x \in E, \|x\| \leq 1\} = \sup\{\|\Phi(\varphi)(x)(y)\| : x \in E, y \in F, \|x\|, \|y\| \leq 1\} = \sup\{\|\varphi(x, y)\| : x \in E, y \in F, \|x\|, \|y\| \leq 1\} \leq \|\varphi\|. \]
5.2. PROJECTIVE TENSOR PRODUCTS OF OPERATOR SPACES

This shows that $\Phi$ maps $BBL(E \times F, G)$ isometrically onto $B(E, B(F, G))$. □

5.2. Projective Tensor Products of Operator Spaces

In this section, we take a review on the definition of projective tensor product of operator spaces and its basic properties. We will see in next section that an important property similar to Theorem 5.1.3 holds for operator spaces (see Theorem 5.3.1).

**Definition 5.2.1.** For operator spaces $V$ and $W$, an operator space matrix norm $\| \cdot \|_\mu$ on $V \otimes W$ is called a subcross-norm (resp. cross norm) if $\| v \otimes w \|_\mu \leq \| v \| \| w \|$ (resp. $\| v \otimes w \|_\mu = \| v \| \| w \|$) for all $v \in M_p(V)$ and $w \in M_q(W)$.

The following lemma can be found in [22].

**Lemma 5.2.2.** Let $V$ and $W$ be operator spaces. Then for any $u \in M_n(V \otimes W)$, there is a decomposition $u = \alpha(v \otimes w)\beta$, where $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{npq}$, $\beta \in M_{pq, n}$, and $p, q \in \mathbb{N}$.

**Theorem 5.2.3.** Let $V$ and $W$ be operator spaces. For any $u \in M_n(V \otimes W)$, define

$$\| u \|_\lambda = \inf \{ \| \alpha \| \| v \| \| w \| \| \beta \| : u = \alpha(v \otimes w)\beta \},$$

where the infimum is taken over all $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{npq}$, $\beta \in M_{pq, n}$ and $p, q \in \mathbb{N}$. Then $\| \cdot \|_\lambda$ is the largest subcross operator space matrix norm on $V \otimes W$.

**Remark 5.2.4.** In fact, $\| \cdot \|_\lambda$ is a cross norm on $V \otimes W$ (see Theorem 6.1.18).

**Proof.** We first prove that $\| \cdot \|_\lambda$ is an operator space matrix norm on $V \otimes W$.

Then we show that this norm is the largest subcross matrix norm on $V \otimes W$.

(1) Given $u_1 \in M_m(V \otimes W)$, $u_2 \in M_n(V \otimes W)$, and $\varepsilon > 0$. By definition, there is a decomposition $u_1 = \alpha_1(v_1 \otimes w_1)\beta_1$ such that $\| u_1 \|_\lambda + \varepsilon \geq \| \alpha_1 \| \| v_1 \| \| w_1 \| \| \beta_1 \|$, where $v_1 \in M_p(V)$, $w_1 \in M_q(W)$, $\alpha_1 \in M_{mpq}$ and $\beta_1 \in M_{pq, m}$. We may assume that $\| v_1 \| = \| w_1 \| = 1$ and $\| \alpha_1 \| = \| \beta_1 \| \leq (\| u_1 \|_\lambda + \varepsilon)^{1/2}$. Similarly, there is a decomposition $u_2 = \alpha_2(v_2 \otimes w_2)\beta_2$ with $\| u_2 \| = \| w_2 \| = 1$ and $\| \alpha_2 \| = \| \beta_2 \| \leq (\| u_2 \|_\lambda + \varepsilon)^{1/2}$. Now,

$$u_1 \otimes u_2 = \begin{pmatrix} \alpha_1(v_1 \otimes w_1)\beta_1 & 0 \\ 0 & \alpha_2(v_2 \otimes w_2)\beta_2 \end{pmatrix}$$

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5.2. PROJECTIVE TENSOR PRODUCTS OF OPERATOR SPACES

\[ (v \otimes w) \begin{pmatrix} v_1 \otimes w_1 & 0 & 0 & 0 \\ 0 & v_1 \otimes w_2 & 0 & 0 \\ 0 & 0 & v_2 \otimes w_1 & 0 \\ 0 & 0 & 0 & v_2 \otimes w_2 \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta_2 \end{pmatrix} \]

where \( v = v_1 \oplus v_2, w = w_1 \oplus w_2 \), and \( \|v\|, \|w\| = 1 \). If we let \( \alpha \) and \( \beta \) be the indicated scalar matrices, then

\[
\|u_1 \oplus u_2\|_\Lambda \leq \|\alpha\| \|\beta\| = \|\alpha \alpha^*\|^{1/2} \|\beta^* \beta\|^{1/2}
\]

\[
= \|\alpha_1 \alpha_1^* \alpha_2 \alpha_2^*\|^{1/2} \|\beta_1 \beta_1^* \beta_2 \beta_2^*\|^{1/2}
\]

\[
= (\max\{\|\alpha_i\|^2\})^{1/2} (\max\{\|\beta_j\|^2\})^{1/2} \leq \max\{\|u_i\|\} + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have M1'.

On the other hand, if \( \gamma \in M_{p,m} \) and \( \delta \in M_{m,p} \), then \( \gamma u_1 \delta = (\gamma \alpha_1)(v_1 \otimes w_1)(\beta_1 \delta) \), and thus

\[
\|\gamma u_1 \delta\|_\Lambda \leq \|\gamma \alpha_1\| \|\beta_1 \delta\| \leq \|\gamma\| \|\delta\| (\|u_1\|_\Lambda + \varepsilon).
\]

Since \( \varepsilon > 0 \) is arbitrary, we have established M2.

By Proposition 4.3.3, in order to show that \( \| \cdot \|_\Lambda \) is an operator space matrix norm, we need to show that, if \( 0 \neq u \in V \otimes W \), then \( \|u\|_\Lambda > 0 \). Suppose that \( u = \sum_{i=1}^n v_i \otimes w_i \neq 0 \). We can assume that the set \( \{w_i : 1 \leq i \leq n\} \) is linearly independent and that \( v_i \neq 0 \) for all \( 1 \leq i \leq n \). We select \( g \in W_{\| \cdot \|_{\leq 1}}^* \) with \( g(w_1) \neq 0 \) and \( g(w_j) = 0 \) for \( j \neq 1 \). Let \( f \in V_{\| \cdot \|_{\leq 1}}^* \) with \( f(v_1) \neq 0 \). Then \( f \otimes g(u) = f(v_1)g(w_1) \neq 0 \).

Now suppose that \( u = \alpha(v \otimes w) \beta \), where \( \alpha \in M_{1,pq}, v \in M_p(V), w \in M_q(W) \) and \( \beta \in M_{pq,1} \). Recall that \( f^{(p)} \) and \( g^{(q)} \) denote the amplification of \( f \) and \( g \), respectively,
and \( \|f^{(p)}\| = \|f\| \leq 1, \|g^{(q)}\| = \|g\| \leq 1. \) So, we have

\[
|(f \otimes g)(u)| = |\alpha((f \otimes g)^{(p)}(v \otimes w))\beta| = |\alpha(f^{(p)}(v) \otimes g^{(q)}(w))\beta|
\leq \|\alpha\| \|f^{(p)}(v)\otimes g^{(q)}(w)\|\|\beta\| = \|\alpha\| \|f^{(p)}(v)\| \|g^{(q)}(w)\| \|\beta\|
= \|\alpha\| \|v\| \|w\| \|\beta\|.
\]

Thus \( \|u\|_{\Lambda} \geq |(f \otimes g)(u)| > 0. \)

(2) Let \( v \in M_p(V), w \in M_q(W), u = v \otimes w \in M_{pq}(V \otimes W). \) Then \( v \otimes w = I_{pq}(v \otimes w)I_{pq}. \) By definition,

\[
\|v \otimes w\|_{\Lambda} \leq \|I_{pq}\| \|v\| \|w\| = \|v\| \|w\|.
\]

Therefore, \( \|\cdot\|_{\Lambda} \) is a subcross norm.

(3) To show that \( \|\cdot\|_{\Lambda} \) is the largest subcross matrix norm on \( V \otimes W, \) let \( \|\cdot\|_{\mu} \) be an arbitrary subcross matrix norm on \( V \otimes W. \) Then for any \( u \in M_n(V \otimes W) \) with decomposition \( u = \alpha(v \otimes w)\beta, \) we have

\[
\|u\|_{\mu} \leq \|\alpha\| \|v \otimes w\| \|\beta\| \leq \|\alpha\| \|v\| \|w\| \|\beta\|,
\]

and thus \( \|u\|_{\mu} \leq \|u\|_{\Lambda}. \)

**Definition 5.2.5.** The norm \( \|\cdot\|_{\Lambda} \) is called the operator space projective tensor product norm. We use \( V \hat{\otimes}_{\Lambda} W \) to denote \( (V \otimes W, \|\cdot\|_{\Lambda}), \) and use \( V \bigotimes_{\Lambda} W \) to denote the completion of \( V \otimes_{\Lambda} W, \) called the operator space projective tensor product of \( V \) and \( W. \)

**Theorem 5.2.6.** Given operator spaces \( V, W \) and \( X, \) we have the completely isometric isomorphisms \( V \hat{\otimes}_{\Lambda} W \cong W \hat{\otimes}_{\Lambda} V \) and \( (V \hat{\otimes}_{\Lambda} W) \hat{\otimes}_{\Lambda} X = V \hat{\otimes}_{\Lambda}(W \hat{\otimes}_{\Lambda} X). \)

**Proof.** Since the proofs of the two identifications are similar, we only prove the first one. It is enough to show that \( V \otimes_{\Lambda} W \cong W \otimes_{\Lambda} V \) as operator spaces. Let \( \varphi : V \otimes_{\Lambda} W \rightarrow W \otimes_{\Lambda} V \) be the unique linear mapping determined by \( \varphi(v \otimes w) = w \otimes v. \) It is easy to see that \( \varphi \) is bijective, and so is \( \varphi^{(n)} \) for each \( n \in \mathbb{N}. \) For any \( u \in M_n(V \otimes_{\Lambda} W), u \) has a representation \( \alpha(v \otimes w)\beta \) for some \( \alpha \in M_{n,pq}, \beta \in M_{pq,n}, v \in M_p(V) \) and \( w \in M_q(W) \) with \( n, p, q \in \mathbb{N}. \) Then

\[
\|\varphi^{(n)}(u)\|_{\Lambda} = \|\alpha\varphi^{(p)}(v \otimes w)\beta\|_{\Lambda} = \|\alpha(w \otimes v)\beta\|_{\Lambda} \leq \|\alpha\| \|v\| \|w\| \|\beta\|.
\]
5.3. COMPLETELY BOUNDED LINEAR MAPPINGS ON PROJECTIVE TENSOR PRODUCTS

So \(\|\varphi^{(n)}(u)\|_\Lambda \leq \|u\|_\Lambda\) for all \(u \in M_n(V \otimes \Lambda W)\). When we interchange \(V\) and \(W\), we get \(\|u\|_\Lambda \leq \|\varphi^{(n)}(u)\|_\Lambda\). Therefore, \(\|\varphi^{(n)}(u)\|_\Lambda = \|u\|_\Lambda\) for all \(n \in \mathbb{N}\) and \(u \in M_n(V \otimes \Lambda W)\). That is, \(\varphi\) is a complete isometry. □

5.3. Completely Bounded Linear Mappings on Projective Tensor Products

Let \(V, W\) and \(X\) be operator spaces. For each bilinear mapping \(\varphi : V \times W \to X\) and \(p, q \in \mathbb{N}\), the \((p; q)\)-th joint amplification of \(\varphi\) is the mapping \(\varphi_{pq} : M_p(V) \times M_q(W) \to M_{pq}(X)\) defined by \(\varphi_{pq}(v, w) = [\varphi(v_{i,j}, w_{k,l})]_{pq} \in M_{pq}(X)\). In case \(p = q\), we shortly write \(\varphi_p\) for \(\varphi_{pq}\). A bilinear mapping \(\varphi : V \times W \to X\) is called jointly completely bounded if \(\|\varphi\|_{\text{cb}} := \sup\{|\varphi_{pq}| : p, q \in \mathbb{N}\} < \infty\). Since for \(p, q \in \mathbb{N}\), \(\|\varphi_{pq}\| \leq \|\varphi_{\max(p,q)}\|\). We see that \(\|\varphi\|_{\text{cb}} = \sup\{|\varphi_p| : p \in \mathbb{N}\}\). We denote the space of all jointly completely bounded bilinear mappings \(\varphi : V \times W \to X\) by \(\text{JCB}(V \times W, X)\). Using the linear space identifications \(M_n(\text{JCB}(V \times W, X)) \cong \text{JCB}(V \times W, M_n(X))\) \((n \in \mathbb{N})\), we can define a matrix norm on \(\text{JCB}(V \times W, X)\). The following theorem shows that, with this matrix norm, \(\text{JCB}(V \times W, X)\) is an operator space.

**Theorem 5.3.1.** For given operator spaces \(V, W\) and \(X\), there are natural isometric identifications

\[
\text{CB}(V \hat{\otimes} W, X) \cong \text{JCB}(V \times W, X) \cong \text{CB}(V, \text{CB}(W, X)).
\]

**Remark 5.3.2.** Replacing \(X\) by \(M_n(X)\) \((n \in \mathbb{N})\), we see that the above identifications are actually completely isometric identifications.

**Proof.** We prove the identifications in the following two parts.

(1) Recall that the natural linear isomorphism \(\Psi : L(V \times W, X) \to L(V \hat{\otimes} W, X)\) is defined by

\[
\Psi(\varphi)(v \otimes w) = \varphi(v, w)
\]

for all \(\varphi \in L(V \times W, X), v \in V\) and \(w \in W\). In the following, we use \(\tilde{\varphi}\) to denote \(\Psi(\varphi)\) for convenience. If \(u = \alpha(v \otimes w)\beta \in M_n(V \hat{\otimes} W)\), where \(v \in M_p(V)\) and \(w \in M_q(W)\), then \(\tilde{\varphi}^{(n)}(u) = \alpha \varphi_{pq}(v, w)\beta \in M_n(X)\). So,

\[
\|\tilde{\varphi}^{(n)}(u)\| = \|\alpha \varphi_{pq}(v, w)\beta\| \leq \|\alpha\| \|\varphi_{pq}\| \|v\| \|w\| \|\beta\| \leq \|\varphi\|_{\text{cb}} \|\alpha\| \|v\| \|w\| \|\beta\|.
\]
This shows that \( \|\varphi^{(n)}(u)\| \leq \|\varphi\|_{\text{cb}}\|u\| \) for all \( u \in M_n(V \otimes W) \) and \( n \in \mathbb{N} \). It implies that \( \|\varphi^{(n)}\| \leq \|\varphi\|_{\text{cb}} \) for all \( n \in \mathbb{N} \). Therefore, \( \|\varphi\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}} \).

Conversely, for all \( p, q \in \mathbb{N} \), we have

\[
\|\varphi_{p,q}\| = \sup\{\|\varphi_{p,q}(v, w)\| : v \in M_p(V), w \in M_q(W), \|v\| \leq 1, \|w\| \leq 1\}
= \sup\{\|\varphi^{(pq)}(v \otimes w)\| : v \in M_p(V), w \in M_q(W), \|v\| \leq 1, \|w\| \leq 1\}
\leq \sup\{\|\varphi^{(pq)}(v \otimes w)\| : v \in M_p(V), w \in M_q(W), \|v \otimes w\|_\Lambda \leq 1\}
\leq \|\varphi^{(pq)}\| \leq \|\varphi\|_{\text{cb}}.
\]

This shows that \( \|\varphi\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}} \). Therefore, \( \|\varphi\|_{\text{cb}} = \|\varphi\|_{\text{cb}} \). So, \( \Psi \) maps \( JCB(V \times W, X) \) isometrically onto \( CB(V \otimes \Lambda W, X) \). Since \( V \otimes \Lambda W \) is dense in \( V \hat{\otimes} W \), each \( \varphi \in CB(V \otimes \Lambda W, X) \) can be uniquely extended to \( V \hat{\otimes} W \). So, we have the isometric identification

\[
CB(V \hat{\otimes} W, X) \cong JCB(V \times W, X).
\]

(2) Recall that the linear isomorphism \( \theta : L(V \times W, X) \to L(V, L(W, X)) \) is defined by

\[
\theta(\varphi)(v)(w) = \varphi(v, w).
\]

We first show that, for given \( \varphi \in JCB(V \times W, X) \) and \( v \in V \), \( \theta(\varphi)(v) \in CB(W, X) \).

Let \( \varphi \in JCB(V \times W, X) \). Then for \( v \in V \) and \( w \in M_q(W) \),

\[
\|\theta(\varphi)(v)\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}} \|v\|,
\]

and thus \( \|\theta(\varphi)(v)\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}} \|v\| \). It follows that \( \theta(\varphi)(V) \subseteq CB(W, X) \).

Next, we show that \( \theta(\varphi) \in CB(V, CB(W, X)) \) and \( \|\theta(\varphi)\|_{\text{cb}} = \|\varphi\|_{\text{cb}} \) for all \( \varphi \in JCB(V \times W, X) \). Let \( v \in M_p(V) \) and \( w \in M_q(W) \). Then

\[
\theta(\varphi)(v)(w) = \varphi_v(w) = \varphi_{v, w} = \varphi_{v, w} = \varphi_{v, w} = \varphi_{v, w},
\]

from which it is evident that \( \|\theta(\varphi)\|_{\text{cb}} = \|\varphi\|_{\text{cb}} \). Therefore, \( \theta \) maps \( JCB(V \times W, X) \) isometrically onto \( CB(V, CB(W, X)) \). □

The following is a direct consequence of Theorem 5.2.6 and Theorem 5.3.1. Since it is very useful, we list it here as a corollary.

**Corollary 5.3.3.** If \( V \) and \( W \) are operator spaces, then we have natural completely isometries.
\[
\lambda : (V \bar{\otimes} W)^* \cong CB(V, W^*) \text{ and } \rho : (V \bar{\otimes} W)^* \cong CB(W, V^*),
\]

where for \( u \in (V \bar{\otimes} W)^*, v \in V \) and \( w \in W \), we have \( (\lambda(u)(v))(w) = (u, v \otimes w) = (\rho(u)(w))(v) \).

**Theorem 5.3.4.** Let \( V, V_1, W, \) and \( W_1 \) be operator spaces, and let \( \phi : V \to V_1 \) and \( \psi : W \to W_1 \) be complete contractions. Then the corresponding mapping \( \phi \otimes \psi : V \otimes W \to V_1 \otimes W_1 \) extends to a complete contraction \( \phi \hat{\otimes} \psi : V \bar{\otimes} W \to V_1 \bar{\otimes} W_1 \).

**Remark 5.3.5.** This theorem holds for completely bounded mappings \( \phi \) and \( \psi \) as well with \( \| \phi \hat{\otimes} \psi \| \leq \| \phi \| \| \psi \| \).

**Proof.** Define \( f : V \times W \to V_1 \otimes W_1 \) by \( f(v, w) = \phi(v) \otimes \psi(w) \). Since \( f \) is bilinear, it corresponds to an \( \bar{f} \in L(V \bar{\otimes} W, V_1 \bar{\otimes} W_1) \) satisfying \( \bar{f}(v \otimes w) = f(v, w) = \phi(v) \otimes \psi(w) \). So \( \bar{f} = \phi \hat{\otimes} \psi \). For \( v \in M_p(V) \) and \( w \in M_q(W) \), we have

\[
\| f_{pq}(v, w) \| = \| \phi(p)(v) \otimes \psi(q)(w) \| \leq \| \phi(p)(v) \| \| \psi(q)(w) \| \leq \| v \| \| w \|.
\]

So, \( f \) is a jointly complete contraction from \( V \times W \) into \( V_1 \bar{\otimes} W_1 \). By Theorem 5.3.1, \( \bar{f} \) is also a complete contraction from \( V \otimes_\Lambda W \) to \( V_1 \otimes_\Lambda W_1 \). Since \( V_1 \bar{\otimes} W_1 \) is complete, \( \bar{f} = \phi \hat{\otimes} \psi \) can be extended to a complete contraction \( \phi \hat{\otimes} \psi : V \bar{\otimes} W \to V_1 \bar{\otimes} W_1 \).

**Remark 5.3.6.** In Theorem 5.3.4, if \( \phi \) and \( \psi \) are complete isometries, then, in general, the mapping \( \phi \hat{\otimes} \psi : V \otimes W \to V_1 \otimes W_1 \) may not be extended to a complete isometry \( \phi \hat{\otimes} \psi : V \bar{\otimes} W \to V_1 \bar{\otimes} W_1 \).

## 5.4. Projectivity

The main purpose of this section is to give a detailed proof of the 'projectivity' of projective tensor products, which is stated in Theorem 5.4.2.

**Definition 5.4.1.** Let \( X \) and \( Y \) be normed spaces. A bounded linear mapping \( \phi : X \to Y \) is called a quotient mapping if the induced mapping \( \tilde{\phi} : X / \ker \phi \to Y \) is a surjective isometry. This is equivalent to saying that \( \phi \) maps \( X_{\| \cdot \| \leq 1} \) onto \( Y_{\| \cdot \| \leq 1} \). Let \( V \) and \( W \) be operator spaces. A bounded linear mapping \( \phi : V \to W \) is called a complete quotient mapping if the amplifications \( \phi^{(n)} \) are quotient mappings for all \( n \in \mathbb{N} \).
Theorem 5.4.2. Let $V, V_1, W,$ and $W_1$ be operator spaces, and $\varphi : V \to V_1$ and $\psi : W \to W_1$ complete quotient mappings. Then $\varphi \otimes \psi : V \otimes W \to V_1 \otimes W_1$ extends to a complete quotient mapping $\varphi \otimes \psi : V \otimes W \to V_1 \otimes W_1$. Furthermore,

$$\ker(\varphi \otimes \psi) = [(\ker \varphi) \otimes W + V \otimes (\ker \psi)].$$

(30)

Proof. (1) Let $\varphi$ and $\psi$ be complete quotient mappings. We want to show that $(\varphi \otimes \psi)^{(n)}$ maps $(M_n(V \otimes W))_{\| \cdot \|_1}^{\| \cdot \|_1}$ onto $(M_n(V_1 \otimes W_1))_{\| \cdot \|_1}^{\| \cdot \|_1}$ for each $n \in \mathbb{N}$. It is easy to see that $(\varphi \otimes \psi)^{(n)}((M_n(V \otimes W))_{\| \cdot \|_1}^{\| \cdot \|_1}) \subseteq (M_n(V_1 \otimes W_1))_{\| \cdot \|_1}^{\| \cdot \|_1}$. We now prove the converse inclusion. Note that $M_n(V \otimes W)$ and $M_n(V_1 \otimes W_1)$ are completions of $M_n(V \otimes_\Lambda W)$ and $M_n(V_1 \otimes_\Lambda W_1)$, respectively. It suffices to show that $(\varphi \otimes \psi)^{(n)}$ maps $(M_n(V \otimes_\Lambda W))_{\| \cdot \|_1}^{\| \cdot \|_1}$ onto $(M_n(V_1 \otimes_\Lambda W_1))_{\| \cdot \|_1}^{\| \cdot \|_1}$. Given $\tilde{u} \in M_n(V_1 \otimes_\Lambda W_1)$ with $\| \tilde{u} \|_1 < 1$, we may suppose that $\tilde{u} = \alpha(\tilde{v} \otimes \tilde{w})\beta$, where $\tilde{v} \in M_p(V_1)$, $\tilde{w} \in M_q(W_1)$, and $\| \alpha \|_1, \| \beta \|_1, \| \tilde{v} \|_1, \| \tilde{w} \|_1 < 1$. Since both $\varphi$ and $\psi$ are complete quotient mappings, there exist $v \in M_p(V)$ and $w \in M_q(W)$ such that $\varphi(v) = \tilde{v}$ and $\psi(w) = \tilde{w}$ with $\| v \|_1 < 1$ and $\| w \|_1 < 1$. Let $u = \alpha(v \otimes w)\beta$. Then we see that $(\varphi \otimes \psi)^{(n)}(u) = \tilde{u}$ and $\| u \|_1 < 1$.

(2) For a Hilbert space $H$ and a subset $A$ of $H$, we use $A^\perp$ to denote the subset of $H$ consisting of those elements $x$ such that $\langle x, y \rangle = 0$ for all $y \in A$, i.e., $A^\perp = \{ x \in H : \langle x, y \rangle = 0, \forall y \in A \}$. In order to obtain equality (30), by bipolar theorem (see [5, Theorem 5.1.8]), it suffices to show that

$$(\ker(\varphi \otimes \psi))^\perp = [(\ker \varphi) \otimes W + V \otimes (\ker \psi)]^\perp.$$  (31)

First, it is easy to see that $[(\ker \varphi) \otimes W + V \otimes (\ker \psi)] \subseteq \ker(\varphi \otimes \psi)$. Thus,

$$(\ker(\varphi \otimes \psi))^\perp \subseteq [(\ker \varphi) \otimes W + V \otimes (\ker \psi)]^\perp.$$

To show the converse inclusion, for $F \in [(\ker \varphi) \otimes W + V \otimes (\ker \psi)]^\perp$, we define $F_1 \in (V_1 \otimes W_1)^*$ by $F_1(v_1 \otimes w_1) = F(v \otimes w)$ in which $v \in V, w \in W, v_1 = \varphi(v)$ and $w_1 = \psi(w)$. We first prove that $F_1$ is well-defined. Suppose that $v' \in V$ and $w' \in W$ satisfy $\varphi(v) = \varphi(v')$ and $\psi(w) = \psi(w')$. Then $v - v' \in \ker \varphi$ and $w - w' \in \ker \psi$. Since $F \in [(\ker \varphi) \otimes W + V \otimes (\ker \psi)]^\perp$, we have $F((v - v') \otimes w) = F(v' \otimes (w - w')) = 0$. Then

$$F(v \otimes w - v' \otimes w') = F(v \otimes (w - w') + (v - v') \otimes w')$$

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\[ F(v \otimes (v - v')) + F((v - v') \otimes w') = 0. \]

Thus, \( F(v \otimes w) = F(v' \otimes w') \). Now, note that \( F(v \otimes w) = F_1 \circ (\varphi \otimes \psi)(v \otimes w) \), and so

\[ F(u) = F_1 \circ (\varphi \otimes \psi)(u) \quad (32) \]

for all \( u \in V \otimes \Lambda W \). Since \( V \otimes \Lambda W \) is dense in \( V \otimes W \), the equality holds for all \( u \in V \otimes W \). It follows that \( F \in (\ker(\varphi \otimes \psi))^\perp \). Thus

\[ [(\ker \varphi) \otimes W + V \otimes (\ker \psi)]^\perp \subseteq (\ker(\varphi \otimes \psi))^\perp. \]

\[ \square \]

5.5. Trace Class on Operator Spaces

In this section, we first take a review on the space \( T_n \) of trace class operators on \( \mathbb{C}^n \) and generalize it to \( T_n(V) \) for an operator space \( V (n \in \mathbb{N}) \). Then we present some properties of \( T_n(V) \).

**Definition 5.5.1.** For any \( \alpha \in \mathbb{M}_{m,n} \), the norm \( \| \cdot \|_\infty \), usually denoted by \( \| \cdot \| \), is the operator norm determined by identifying \( \mathbb{M}_{m,n} \) with \( B(\mathbb{C}^n, \mathbb{C}^m) \); \( \| \cdot \|_2 \) is the Hilbert-Schmidt norm defined by \( \| \alpha \|_2 = \sqrt{\text{trace}(\alpha \alpha^*)} = \left( \sum_{i,j=1}^{m,n} |\alpha_{ij}|^2 \right)^{1/2} \); and \( \| \cdot \|_1 \) is the trace class norm defined by \( \| \alpha \|_1 = \text{trace}(|\alpha|) \), where \( |\alpha| = (\alpha \alpha^*)^{1/2} \). We write \( \mathbb{M}_{m,n}, \mathbb{H} \mathbb{S}_{m,n} \), and \( T_{m,n} \) for the linear space \( \mathbb{M}_{m,n} \) with the norm \( \| \cdot \|_1 \), \( \| \cdot \|_2 \), and \( \| \cdot \|_1 \), respectively. In particular, we let \( M_n = \mathbb{M}_{n,n}, \mathbb{H} \mathbb{S}_n = \mathbb{H} \mathbb{S}_{n,n}, \) and \( T_n = T_{n,n} \).

**Lemma 5.5.2.** For any \( \alpha \in \mathbb{M}_{n,r} \) and \( n, r \in \mathbb{N} \), we have \( \| \alpha \| \leq \| \alpha \|_2 \).

**Proof.**

\[ \| \alpha \|^2 = \sup_{\| x \|_2 \leq 1} \| \alpha x \|^2_2 \]

\[ = \sup_{\| x \|_2 \leq 1} \left\| \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_2^2 \]

\[ = \sup_{\| x \|_2 \leq 1} \left\| \begin{pmatrix} \alpha_{11} x_1 + \cdots + \alpha_{1n} x_n \\ \vdots \\ \alpha_{m1} x_1 + \cdots + \alpha_{mn} x_n \end{pmatrix} \right\|_2^2 \]

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\[ \sup_{\|x\| \leq 1} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |a_{ij}x_j| \right)^2 \leq \sup_{\|x\| \leq 1} \left( \sum_{j=1}^{n} |\alpha_{ij}|^2 \right) \left( \sum_{j=1}^{n} |x_j|^2 \right) \]
\[ \leq \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |\alpha_{ij}|^2 \right) = \|\alpha\|^2. \]

So, \( \|a\| \leq \|a\|^2. \)

**Theorem 5.5.3.** Let \( V \) be an operator space and \( n \in \mathbb{N} \). For \( v \in M_n(V) \), define
\[ \|v\|_1 = \inf \{ \|\alpha\|_2 \|\beta\|_2 : v = \alpha \tilde{v} \beta, \alpha \in HS_n, \beta \in HS_n, \tilde{v} \in M_n(V), r \in \mathbb{N} \}. \]

Then \( \|\cdot\|_1 \) is a norm on \( M_n(V) \) satisfying \( \|v\| \leq \|v\|_1 \leq n\|v\| \). We denote by \( T_n(V) \) the normed space \( (M_n(V), \|\cdot\|_1) \).

**Proof.** Given \( v_1, v_2 \in M_n(V) \) and \( \varepsilon > 0 \), let \( v_i = \alpha_i \tilde{v}_i \beta_i \) with \( \|\tilde{v}_i\| \leq 1 \) and \( \|\alpha_i\|_2 = \|\beta_i\|_2 < (\|v_i\|_1 + \varepsilon)^{1/2} \). Let \( \alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2]^t \), and \( \tilde{v} = \tilde{v}_1 \oplus \tilde{v}_2 \). Then
\[ \|\alpha\|_2^2 = \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2, \|\beta\|_2^2 = \|\beta_1\|_2^2 + \|\beta_2\|_2^2, \text{ and } \|\tilde{v}\| \leq 1. \]

Note that \( v_1 + v_2 = \alpha \tilde{v} \beta \), so
\[ \|v_1 + v_2\|_1 \leq \|\alpha\|_2 \|\beta\|_2 \leq (\|\alpha\|_2^2 + \|\beta\|_2^2)/2 \]
\[ = (\|\alpha_1\|_2^2 + \|\beta_1\|_2^2 + \|\alpha_2\|_2^2 + \|\beta_2\|_2^2)/2 \leq \|v_1\|_1 + \|v_2\|_1 + 2\varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we get the subadditivity. For any \( c \in \mathbb{C} \), we have \( cv_1 = \alpha (c \tilde{v}) \beta \), and hence \( \|cv_1\|_1 \leq |c| \|v_1\|_1 \). The inequality \( \|\cdot\|_1 \leq \|\cdot\|_1 \) will imply that \( \|\cdot\|_1 \) is a norm. To prove the above inequality, let \( v \in M_n(V) \) and \( v = \alpha \tilde{v} \beta \). By Lemma 5.5.2, \( \|v\| \leq \|\alpha\| \|\tilde{v}\| \|\beta\|_2 \). Thus, \( \|v\|_1 \leq \|v\|_1 \).

Finally, since \( v = I_n v I_n \), it follows that \( \|v\|_1 \leq \|I\|_2 \|v\| \|I\|_2 = n \|v\| \).

**Lemma 5.5.4.** Let \( V \) be an operator space and \( v \in M_n(V) \) \( (n \in \mathbb{N}) \). Then \( \|v\|_1 < 1 \) if and only if there exist \( \alpha \in M_{n,r}, \tilde{v} \in M_r(V) \) and \( \beta \in M_{r,n} \) for some \( r \in \mathbb{N} \) such that \( v = \alpha \tilde{v} \beta \) and \( \|\alpha\|_2, \|\tilde{v}\|, \|\beta\|_2 < 1 \).

**Proof.** Suppose that \( v \in M_n(V) \) with \( \|v\| < 1 \). Then there exist \( \alpha \in M_{n,r}, \beta \in M_{r,n} \) and \( \tilde{v} \in M_r(V) \) for some \( r \in \mathbb{N} \) such that \( v = \alpha \tilde{v} \beta \) and \( \|\alpha\| \|\tilde{v}\| \|\beta\| < 1 \). For \( \varepsilon > 0 \), let
\[ c_\varepsilon = (\|\alpha\| + \varepsilon)(\|\tilde{v}\| + \varepsilon)(\|\beta\| + \varepsilon), \]

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\[ \alpha' = \frac{c_\varepsilon^{1/3}}{\|\alpha\| + \varepsilon}, \quad \beta' = \frac{c_\varepsilon^{1/3}}{\|\beta\| + \varepsilon}, \quad \text{and} \quad \beta' = \frac{c_\varepsilon^{1/3}}{\|\beta\| + \varepsilon}. \]

Then \( \alpha'\beta' = \alpha\beta = v \). Moreover,

\[ \|\alpha'\| < c_\varepsilon^{1/3}, \quad \|\beta'\| < c_\varepsilon^{1/3}, \quad \text{and} \quad \|\beta'\| < c_\varepsilon^{1/3}. \]

Since \( \|\alpha\|\|\beta\| < 1 \), we can choose \( \varepsilon \) small enough so that \( c_\varepsilon^{1/3} < 1 \). Then the decomposition \( v = \alpha'\beta' \) with the chosen \( \varepsilon \) is just what we want.

The other direction of the implication follows from the definition of \( \| \cdot \|_1 \). \( \square \)

The following theorem reveals the dual relationships between \( M_n(V) \) and \( T_n(V^*) \), and between \( T_n(V) \) and \( M_n(V^*) \). Although this theorem only shows the isometric identifications, they are actually complete isometric identifications (see Theorem 5.5.8).

**Theorem 5.5.5.** Let \( V \) be an operator space. The scalar pairing \( \langle \cdot, \cdot \rangle : M_{m,n}(V) \times M_{m,n}(V^*) \to \mathbb{C} \) defined by \( \langle v, w \rangle = \sum (v_{ij}, w_{ij}) \) determines the isometric identifications

\[ T_n(V)^* \cong M_n(V^*) \text{ and } M_n(V^*) \cong T_n(V^*). \]

**Proof.** Define \( \Psi : M_n(V^*) \to T_n(V)^* \) by \( \Psi(f)(v) = \langle v, f \rangle \) \( (f \in M_n(V^*)) \) and that \( \Psi \) is surjective and isometric.

For any \( F \in T_n(V)^* \) and \( v = (v_{ij}) \in T_n(V) \), \( F(v) = F(\sum A_{ij}(v_{ij})) = \sum F(A_{ij}(v_{ij})) \), where \( A_{ij}(x) \) denote the \( n \times n \) matrix with value \( x \) at \((i, j)\)-th position and values 0 at other positions. Let \( f_{ij}(x) = F(A_{ij}(x)) \) for all \( x \in V \). Then \( f_{ij} \in V^* \) and \( f = (f_{ij}) \in M_n(V^*) \). It is easy to see that \( \Psi(f) = F \). Thus, \( \Psi \) is surjective.

Recall that we have

\[ \|f\| = \sup\{\|\langle f, \tilde{v} \rangle\| : \tilde{v} \in M_r(V), \|\tilde{v}\| \leq 1, r \in \mathbb{N} \} \]

for all \( f \in M_n(V^*) \). Note that \( \langle f, \tilde{v} \rangle \in M_{nr} \). So

\[ \|\langle f, \tilde{v} \rangle\| = \sup\{\|\langle f, \tilde{v} \rangle \eta \xi\| : \xi, \eta \in D_{nr} \}, \]

where \( D_{nr} \) is the unit ball of \( \mathbb{C}^{nr} \). For any \( x = (x_m)_{m=1}^\infty \in D_{nr} \), we can rewrite \( (x_m) = (x_{ij}) \) by rearranging the sub-indices such that \( m = (i-1) \times r + j, (1 \leq i \leq n, 1 \leq j \leq n) \).
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Then

\[ ||f|| = \sup \{ ||\langle (f, \bar{v}) \rangle \xi \eta || : \xi, \eta \in D_{nr}, \bar{v} \in M_r(V), ||\bar{v}|| \leq 1, r \in \mathbb{N} \} \]

\[ = \sup \{ \sum_{i,j,k,l} f_{ki}(\bar{u}_{ij}) \eta j \xi ik : \xi, \eta \in D_{nr}, \bar{v} \in M_r(V), ||\bar{v}|| \leq 1, r \in \mathbb{N} \} \]

\[ = \sup \{ \sum_{k,l} \langle f_{ki}, \sum_{i,j} \bar{\xi}_{ik} \bar{u}_{ij} \eta j \rangle : \xi, \eta \in D_{nr}, \bar{v} \in M_r(V), ||\bar{v}|| \leq 1, r \in \mathbb{N} \}. \]

Let \( \alpha_{ki} = \bar{\xi}_{ik} \) and \( \beta_{ji} = \eta_{ji} \). Then \( \alpha = [\alpha_{ki}] \in HS_{n,r}, \beta = [\beta_{ji}] \in HS_{r,n} \), and

\[ ||\alpha||_2, ||\beta||_2 \leq 1. \]

We have

\[ ||f||_{M_n(V^*)} = \sup \{ \sum_{k,l} \langle f_{ki}, (\alpha \bar{v} \beta)_{kl} \rangle : \bar{v} \in M_r(V), ||\bar{v}|| \leq 1, r \in \mathbb{N}, ||\alpha||_2 \leq 1, ||\beta||_2 \leq 1 \} \]

\[ = \sup \{ \langle f, \alpha \bar{v} \beta \rangle : \bar{v} \in M_r(V), ||\bar{v}|| \leq 1, r \in \mathbb{N}, ||\alpha||_2 \leq 1, ||\beta||_2 \leq 1 \} \]

\[ = \sup \{ \langle f, v \rangle : v = \alpha \bar{v} \beta, \bar{v} \in M_r(V), ||\bar{v}|| \leq 1, r \in \mathbb{N}, ||\alpha||_2 \leq 1, ||\beta||_2 \leq 1 \} \]

\[ = \sup \{ \langle f, v \rangle : v \in M_n(V), ||v||_1 \leq 1 \} = ||\Psi(f)||_{T_n(V^*)}. \]

Thus, \( \Psi \) is isometric. The forth equality holds due to Lemma 5.5.4. For completeness, we include its proof here. Let

\[ A = \sup \{ \langle f, v \rangle : v \in M_n(V), ||v||_1 \leq 1 \} \]

\[ B = \sup \{ \langle f, v \rangle : v = \alpha \bar{v} \beta, \bar{v} \in M_r(V), \alpha \in M_{n,r}, \beta \in M_{r,n}, ||\bar{v}||_1, ||\alpha||_2, ||\beta||_2 \leq 1, r \in \mathbb{N} \}. \]

We want to show that \( A = B \).

Clearly, by the definition of \( || \cdot ||_1 \), \( B \leq A \).

Now, let \( v \in M_n(V) \) with \( ||v||_1 \leq 1 \). Then \( ||v||_1 < 1 + \varepsilon \), or \( \left| \frac{v}{1 + \varepsilon} \right|_1 < 1 \) for any \( \varepsilon > 0 \). By Lemma 5.5.4, there exists a decomposition of \( \frac{v}{1 + \varepsilon} = \alpha' \bar{v}' \beta' \) with \( ||\alpha'||_2, ||\bar{v}'||, ||\beta'||_2 < 1 \). So \( \left| \langle f, \frac{v}{1 + \varepsilon} \rangle \right| \leq B \), or \( ||\langle f, v \rangle|| \leq (1 + \varepsilon)B \). Since \( \varepsilon \) is arbitrary, we have \( ||\langle f, v \rangle|| \leq B \). Taking supremum over all \( v \in M_n(V) \) with \( ||v||_1 \leq 1 \), we get \( A \leq B \). Thus \( A = B \).

To show the second identification, let \( \Psi : T_n(V^*) \to M_n(V^*) \) be the same linear mapping as above. Given a functional \( F \in M_n(V^*) \) with \( ||F|| < 1 \), there exists a complete contraction \( \tilde{f} : V \to M_n \) and vectors \( \xi, \eta \in (\mathbb{C}^n)^n \cong \mathbb{C}^{n^2} \) with \( ||\xi|| < 1, ||\eta|| < 1 \) such that \( F(v) = \langle f(\alpha)(v) \eta | \xi \rangle \) for all \( v \in M_n(V) \) (see [13, Lemma 2.3.3]). With the identification \( CB(V, M_n) \cong M_n(V^*) \), we can write \( \tilde{f} \) as \( (\tilde{f}_{kl}) (1 \leq k, l \leq n) \).
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So
\[ \langle F, v \rangle = \langle \tilde{f}^{(n)}(v) \eta | \xi \rangle = \sum_{i,j,k,l} \tilde{f}_{kl}(v_{ij}) \eta_{jl} \xi_{ik} = \sum_{i,j} \left[ \sum_{k,l} \tilde{f}_{kl} \eta_{jl}(v_{ij}) \right]. \]

Let \( f_{ij} = \sum_{k,l} \tilde{f}_{kl} \eta_{jl} \) and \( f = (f_{ij}) \). Then \( f \in T_n(V^*) \). It is easy to see that \( F = \Psi(f) \). Thus \( \Psi \) is surjective. Moreover, \( f = \alpha \tilde{f} \beta \), where \( \alpha_{ik} = \tilde{f}_{ik} \) and \( \beta_{ij} = \eta_{ij} \) satisfying \( ||\alpha||_2 < 1 \) and \( ||\beta||_2 < 1 \). Thus \( ||f||_1 \leq ||\alpha||_2 ||\tilde{f}||_1 ||\beta||_2 < 1 \). It follows that \( ||f||_1 \leq ||\Psi(f)|| \).

Conversely, for any \( f \in T_n(V^*) \), if \( ||f||_1 < 1 \), then there exists a decomposition \( f = \alpha \tilde{f} \beta \) with \( ||\tilde{f}||_{cb} < 1 \) and \( ||\alpha||_2, ||\beta||_2 < 1 \). Let \( \eta \in \mathbb{C}^n \) and \( \xi \in \mathbb{C}^n \) be the vectors corresponding to \( \alpha \) and \( \beta \) as above, respectively. Thus, for all \( v \in M_n(V) \),

\[ ||\Psi(f)(v)|| = ||(f,v)|| = ||(\alpha \tilde{f} \beta, v)|| = ||(\langle \tilde{f}, v \rangle) \eta | \xi \rangle|| \leq ||\langle \tilde{f}, v \rangle||_1 ||\eta||_1 ||\xi||_1 \leq ||\langle \tilde{f}, v \rangle||. \]

By Equality (25), we have

\[ ||\Psi(f)(v)|| \leq ||\langle \tilde{f}, v \rangle|| \leq ||v||. \]

This shows that \( ||\Psi(f)||_{M_n(V)} \leq ||f||_1 \). Therefore, \( ||\Psi(f)||_{M_n(V)} = ||f||_1 \). \( \Box \)

Given operator spaces \( V \) and \( W \), let \( \varphi : V \longrightarrow W \) be a linear mapping and \( id : M_n \longrightarrow M_n \) \((n \in \mathbb{N})\) be the identity mapping. Let \( \varphi^{(n)} = id \otimes \varphi \) be the n-th amplification of \( \varphi \) from \( M_n(V) \) to \( M_n(W) \). We denote by \( T_n(\varphi) = id \otimes \varphi \) the n-th amplification of \( \varphi \) from \( T_n(V) \) to \( T_n(W) \). Note that we have \( T_n(\varphi^*) = (id \otimes \varphi^*) = id \otimes (\varphi^*)^{(n)} \). Similarly, we have \( (\varphi^{(n)})^* = T_n(\varphi^*) \). The following theorem and its corollary can be found in [13, Proposition 4.1.8 and Corollary 4.1.9].

**Theorem 5.5.6.** Given operator spaces \( V \) and \( W \) and a linear mapping \( \varphi : V \longrightarrow W \), \( ||T_n(\varphi)|| = ||\varphi^{(n)}|| \) and the following are equivalent.

(a) \( \varphi^{(n)} \) is an isometric injection.
(b) \( T_n(\varphi) \) is an isometric injection.
(c) \( (\varphi^*)^{(n)} \) is a quotient mapping.
(d) \( T_n(\varphi^*) \) is a quotient mapping.

If, furthermore, \( V \) is complete, then the following are equivalent.

(a) \( \varphi^{(n)} \) is a quotient mapping.
(b) \( T_n(\varphi) \) is a quotient mapping.
(c) \((\varphi^*)^{(n)}\) is an isometric injection.

(d) \(T_n(\varphi^*)\) is an isometric injection.

**Corollary 5.5.7.** Let \(V\) and \(W\) be operator spaces and \(\varphi : V \rightarrow W\) be a linear mapping. Then \(\varphi\) is a complete isometry if and only if \(\varphi^* : W^* \rightarrow V^*\) is an exact complete quotient mapping. If \(V\) and \(W\) are complete, then \(\varphi : V \rightarrow W\) is a complete quotient mapping if and only if \(\varphi^*\) is a complete isometry. In the latter case, \(\varphi^*(W^*)\) is weak* closed, and \(\varphi^*\) is a weak* homeomorphism in the topologies defined by \(V\) and \(W\), respectively.

**Theorem 5.5.8.** Let \(n, r \in \mathbb{N}\). Then we have complete isometry \(T_n \hat{\otimes} T_r \cong T_{n \times r}\).

If \(V\) is a complete operator space, then we have the following completely isometric identifications

\[- T_n(V) \cong T_n \hat{\otimes} V, \quad T_n(V)^* \cong M_n(V^*), \]
\[- M_n(V)^* \cong T_n(V^*), \quad \text{and} \quad M_n(V^{**}) \cong M_n(V^{**}). \]

**Proof.** By Corollary 5.3.3, we have the complete isometries

\[- (V \hat{\otimes} W)^* \cong CB(V, W^*) \cong CB(W, V^*). \]

Since \(T_n^* = M_n\) as operator spaces (see [13, Theorem 3.2.3]), if we replace \(V\) and \(W\) above by \(T_n\) and \(V\), respectively, we get the following complete isometry

\[- (T_n \hat{\otimes} V)^* \cong CB(V, M_n) = M_n(V^*). \] (33)

Let \(V = T_r\). Then \((T_n \hat{\otimes} T_r)^* \cong M_n(M_r) = M_{n \times r}\), and thus

\[- T_n \hat{\otimes} T_r \cong (T_n \hat{\otimes} T_r)^{**} \cong M_{n \times r}^* \cong T_{n \times r}. \]

Combining equality (33) and Theorem 5.5.5, we get a natural isometry \(T_n(V)^* \cong (T_n \hat{\otimes} V)^*\). Note that \(T_n(V) \cong T_n \hat{\otimes} V\) as linear spaces and the norms on \(T_n(V)^*\) and \(T_n \hat{\otimes} V\) are determined by \(T_n(V)^*\) and \((T_n \hat{\otimes} V)^*\), respectively. Thus

\[- T_n(V) \cong T_n \hat{\otimes} V \] (34)

as normed spaces.

We use equality (34) to define an operator space matrix norm on \(T_n(V)\). Then equality (34) is a complete isometry. Combining equalities (33) and (34), we have the...
following complete isometries
\[ T_n(V)^* = (T_n \hat{\otimes} V)^* \cong CB(V, M_n) \cong M_n(V^*). \]

From Theorem 5.5.5 and equality (34), we also have a natural isometry
\[ T_n(V^*) \cong T_n \hat{\otimes} V^* \cong M_n(V)^*. \]

In order to show that it is also a complete isometry, it suffices to show that, for each
\( s \in \mathbb{N}, \) we have the corresponding isometry \( M_s(T_n(V^*)) \cong M_n(M_n(V)^*). \) By Theorem 5.5.6, this is equivalent to showing that \( T_s(T_n(V^*)) \cong T_s(M_n(V)^*) \) isometrically for all \( s \in \mathbb{N}. \) This is true because we have the following isometries:
\[ T_s(T_n(V^*)) \cong T_s \hat{\otimes} T_n \hat{\otimes} V^* \cong T_{s \times n} \hat{\otimes} V^* \]
\[ \cong T_{s \times n}(V^*) \cong M_{s \times n}(V)^* \text{ (by Theorem 5.5.5)} \]
\[ \cong [M_s(M_n(V))]^* \cong T_s(M_n(V)^*) \text{ (by Theorem 5.5.5)}. \]

The last identification is trivial. \( \square \)

5.6. Fubini Theorem

Let \( H \) and \( K \) be Hilbert spaces. The main purpose of this section is to prove the Fubini theorem \( B(H)^* \hat{\otimes} B(K)^* \cong B(H \otimes_2 K)^*. \) Or equivalently, \( TC(H)^* \hat{\otimes} TC(K) \cong TC(H \otimes_2 K). \)

Although we have operator space identifications
\[ M_n(B(H)) \cong B(H^n) \text{ and } M_n(K(H)) \cong K(H^n), \]
we do not have in general the operator space identification \( M_n(TC(H)) \cong TC(H^n). \)

This is because
\[ TC(H^n) \cong K(H^n)^* \cong M_n(K(H))^* \cong T_n(K(H)^*) \cong T_n(TC(H)) \]
and \( T_n(TC(H)) \neq M_n(TC(H)). \) For example, if we take \( H = \mathbb{C}, \) then \( TC(H) = TC(\mathbb{C}) = \mathbb{C}. \) So, \( T_n(TC(H)) = T_n(TC(\mathbb{C})) = T_n \text{ and } M_n(TC(H)) = M_n(TC(\mathbb{C})) = M_n. \) But, \( T_n \neq M_n \) if \( n \geq 2. \)

Lemma 5.6.1. Let \( H_0 \) and \( K_0 \) be Hilbert subspaces of \( H \) and \( K, \) respectively. Then there are natural operator space embeddings
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\[ TC(H_0) \hookrightarrow TC(H) \text{ and } TC(H_0) \hat{\otimes} TC(K_0) \hookrightarrow TC(H) \hat{\otimes} TC(K) \]

PROOF. Let \( e \) be the orthogonal projection of \( H \) onto \( H_0 \). Define \( p : K(H) \to K(H_0) \) by \( p(x) = e \circ x|_{H_0} \) for all \( x \in K(H) \). To see that \( p(x) \in K(H_0) \), let \( b_H \) and \( b_{H_0} \) denote the unit balls of \( H \) and \( H_0 \), respectively. Note that we have \( (e \circ x|_{H_0})|_{b_{H_0}} \subseteq \overline{x(b_H)} \). The right hand side is a compact set since \( x \) is compact, so the left hand side is also a compact set. This shows that \( p(x) \) is compact.

Now look at the dual mapping \( p^* : TC(H_0) \to TC(H) \), we have \( \langle p^*(t), x \rangle = \langle t, p(x) \rangle = \langle t, e \circ x|_{H_0} \rangle \). We want to show that \( p^* \) is completely isometric. For any \( t \in TC(H_0) \),

\[
\|p^*(t)\| = \sup\{|\langle p^*(t), x \rangle| : x \in K(H), \|x\| \leq 1\}
= \sup\{|\langle t, e \circ x|_{H_0} \rangle| : x \in K(H), \|x\| \leq 1\}
= \sup\{|\langle t, x_0^0 \rangle| : x_0^0 \in K(H_0), \|x_0^0\| \leq 1\} = \|t\|.
\]

Thus, \( p \) is isometric. For any \( T \in M_n(TC(H_0)) \) \((n \in \mathbb{N})\), \( (p^*)^{(n)}(T) \in M_n(TC(H)) \cong CB(K(H), M_n) \). So, by [13, Proposition 2.2.2], we have \( \|((p^*)^{(n)}(T))_{ob}\| = \|((p^*)^{(n)}(T))^{(n)}\| \).

Note that \( ((p^*)^{(n)}(T))^{(n)} : M_n(K(H)) \to (M_n)_n = M_{n^2} \). So

\[
\|((p^*)^{(n)}(T))^{(n)}\| = \sup\{|\langle (p^*)^{(n)}(T)^{(n)}(x), x \rangle| : x \in M_n(K(H)), \|x\| \leq 1\}
= \sup\{|\langle (p^*)^{(n)}(T), x_{ij} \rangle| : x \in M_n(K(H)), \|x\| \leq 1\}
= \sup\{|\langle (p^*(T_{kh}), x_{ij}) \rangle| : x \in M_n(K(H)), \|x\| \leq 1\}
= \sup\{|\langle (T_{kh}, x_{ij}) \rangle| : x \in M_n(K(H)), \|x\| \leq 1\}
= \sup\{|\langle (T_{kh}, e \circ x|_{H_0}) \rangle| : x \in M_n(K(H)), \|x\| \leq 1\}
= \sup\{|\langle (T_{kh}, x_{ij}) \rangle| : x_0 \in M_n(K(H_0)), \|x_0\| \leq 1\}
= \|T^{(n)}\| = \|T\|_{ob}.
\]

This shows that \( (p^*)^{(n)} \) are isometric for all \( n \in \mathbb{N} \). That is, \( p^* : TC(H_0) \to TC(H) \) is completely isometric.

Note that in the above arguments on \( \|p^*(t)\| \) and \( \|((p^*)^{(n)}(T))^{(n)}\| \), we used the fact that, for all \( n \in \mathbb{N} \),

\[
\{(e \circ x|_{H_0}) : x \in M_n(K(H)), \|x\| \leq 1\} = \{x_{ij}^0 : x_0 \in M_n(K(H_0)), \|x_0\| \leq 1\}.
\]
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We show here that the equality is true. Note that $M_n(K(H)) \cong K(H^n) \subseteq B(H^n)$ and $M_n(K(H_0)) \cong K(H_0^n) \subseteq B(H_0^n)$. Let $P : B(H) \rightarrow B(H_0)$ be the canonical projection. Then $P^{(n)} : B(H^n) \rightarrow B(H_0^n)$.

It is easy to see that $\|P^{(n)}\| = 1$ and $P^{(n)}|_{K(H^n)} = p^{(n)}$. So, $\|p^{(n)}\| \leq 1$ and thus $p^{(n)}(M_n(K(H_0))) \subseteq M_n(K(H_0))$.

On the other hand, for any $x^0 \in M_n(K(H_0))$ with $\|x^0\| \leq 1$, define $x = (x_{ij}) \in M_n(B(H))$ by $x_{ij}(h) = x^0_{ij} \circ e(h) = x^0_{ij}(h_0)$, where $h = h_0 + h_0^\perp \in H_0 + H_0^\perp = H$.

Then $x \in M_n(K(H))$. To see this, note that

$$x_{ij}(b_H) = \{x_{ij}(h) : h \in b_H\} = \{x^0_{ij}(h_0) : h_0 \in H_0, h = h_0 + h_0^\perp \in b_H\}
= \{x^0_{ij}(h_0) : h_0 \in b_{H_0}\} = x^0_{ij}(b_{H_0}).$$

So, $x_{ij}(b_H) = x^0_{ij}(b_{H_0})$. Then $\overline{x_{ij}(b_H)} = \overline{x^0_{ij}(b_{H_0})}$. Thus $x^0_{ij}$ is compact implies that $x_{ij}$ is compact for $1 \leq i, j \leq n$. So, $x$ is in $M_n(K(H))$. Moreover, we have

$$\|x\| = \sup\{\|x(h)\| : h \in H^n, \|h\| \leq 1\}
= \sup\{\|x^0(h_0)\| : h_0 \in H_0^n, \|h_0\| \leq 1\} = \|x^0\| \leq 1.$$}

Now, $x_0 = p^{(n)}(x)$. Therefore, $M_n(K(H_0)) \subseteq p^{(n)}(M_n(K(H)))$.

To see that the second embedding in the lemma is completely isometric, it suffices to show that

$$TC(H_0) \otimes \Lambda TC(K_0) \rightarrow TC(H) \otimes \Lambda TC(K)$$

is a completely isometric embedding. Now we write $e_H$ for the orthogonal projection from $H$ onto $H_0$ and $e_K$ the orthogonal projection from $K$ onto $K_0$, respectively.

Also, $p_H : K(H) \rightarrow K(H_0)$ and $p_K : K(K) \rightarrow K(K_0)$ denote the mappings $x \mapsto e_H \circ x|_{H_0}$ and $y \mapsto e_K \circ y|_{K_0}$, respectively. By the above discussion, both $p_H^*$ and $p_K^*$ are completely isometric. Define

$$\Phi : TC(H_0) \otimes \Lambda TC(K_0) \rightarrow TC(H) \otimes \Lambda TC(K)$$

by $\Phi = p_H^* \otimes p_K^*$. By Theorem 5.3.4, $\Phi$ is a complete contraction.

In the following, we want to prove that each $\Phi^{(n)}$ is isometric. Define $j_H : K(H_0) \rightarrow K(H)$ by $j_H(x) = x \circ e_H$. Then $j_H$ is induced by the natural completely isometric embedding $B(H_0) \hookrightarrow B(H)$. So, $j_H$ is completely isometric and
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hence \( j^*_H \) is a complete contraction. The same is true for \( j_K : K(K_0) \to K(K) \). By Theorem 5.3.4, the mapping

\[
\Psi : TC(H) \otimes_{\Lambda} TC(K) \to TC(H_0) \otimes_{\Lambda} TC(K_0)
\]

given by \( \Psi = j^*_H \otimes j^*_K \) is a complete contraction. Note that, for all \( x \in K(H_0), e_H \circ (x \circ e_H)|H_0 = x \), i.e., \( p_H \circ j_H = id \). Thus, \( j^*_H \circ p^*_H = id \). Similarly, \( j^*_K \circ p^*_K = id \). So, \( \Psi \circ \Phi = id \). Therefore, for all \( u \in M_n(TC(H_0) \otimes_{\Lambda} TC(K_0)) \),

\[
||u|| = ||\Psi(n)(\Phi(n)(u))|| \leq ||\Phi(n)(u)|| \leq ||u||.
\]

It follows that each \( \Phi(n) \) is isometric. □

Now, we prove the following Fubini Theorem.

**Theorem 5.6.2.** For any Hilbert spaces \( H \) and \( K \), we have a natural complete isometry

\[ B(H)_* \hat{\otimes} B(K)_* \cong B(H \otimes_2 K)_*. \]

**Proof.** Let \( H_0 \) and \( K_0 \) be finite dimensional Hilbert subspaces of \( H \) and \( K \), respectively. Suppose that \( \text{dim}(H_0) = n \) and \( \text{dim}(K_0) = r \). Then \( \text{dim}(H_0 \otimes K_0) = nr \).

So, \( TC(H_0) \cong T_n, TC(K_0) \cong T_r \) and \( TC(H_0 \otimes K_0) \cong T_{nr} \). By Theorem 5.5.8, we have \( TC(H_0) \hat{\otimes} TC(K_0) \cong TC(H_0 \otimes K_0) \). Now, we have a commutative diagram

\[
\begin{array}{ccc}
TC(H_0) \hat{\otimes} TC(K_0) & \to & TC(H_0 \otimes K_0) \\
\downarrow & & \downarrow \\
TC(H) \hat{\otimes} TC(K) & \to & TC(H \otimes_2 K)
\end{array}
\]

By the above argument, the top row is a complete isometry. By Lemma 5.6.1, both columns are completely isometric. Note that every operator \( T \in TC(H) \) has a form

\[
\sum_{n=1}^{\infty} \alpha_n \omega_{\xi_n, \eta_n}
\]

with \( ||T|| = \sum_{n=1}^{\infty} |\alpha_n| \), where \( \{\alpha_n\} \in l^1 \), \( \{\xi_n\} \) and \( \{\eta_n\} \) are two orthogonal systems in \( H \), and \( \omega_{\xi_n, \eta_n} \) is the rank-one operator defined by \( \omega_{\xi_n, \eta_n} \xi = (\xi | \eta_n) \xi_n \) \((n \in \mathbb{N}) \) (see Theorem 3.3.2). So, the union of the spaces \( TC(H_0) \) is norm dense in \( TC(H) \), the union of spaces \( TC(K_0) \) is norm dense in \( TC(K) \), and the union of the spaces \( TC(H_0) \hat{\otimes} TC(K_0) \) is norm dense in \( TC(H) \hat{\otimes} TC(K) \). Similarly, the union of the
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spaces \( TC(H_0 \otimes K_0) \) is norm dense in \( TC(H \otimes_2 K) \). Since the mappings in the top row are coherent, they determine a complete isometry in the bottom row. So, \( B(H) \hat{\otimes} B(K) \ast \cong B(H \otimes_2 K) \ast \). □

Given a functional \( \omega_1 \in B(H) \ast \), we can define a linear mapping \( \omega_1 \otimes id : B(H) \otimes B(K) \rightarrow B(K) \) by \( (\omega_1 \otimes id)(a \otimes b) = \omega_1(a)b \) for all \( a \in B(H) \) and \( b \in B(K) \). If \( u = a \otimes b \) and \( \omega_2 \in B(K) \ast \), then \( \omega_2((\omega_1 \otimes id)(u)) = \omega_1(a)\omega_2(b) = \langle a \otimes b, \omega_1 \otimes \omega_2 \rangle = \langle u, \omega_1 \otimes \omega_2 \rangle \). By linearity, for any \( u \in B(H) \otimes B(K) \), we have \( \omega_2((\omega_1 \otimes id)(u)) = \langle u, \omega_1 \otimes \omega_2 \rangle \) for all \( u \in B(H) \otimes B(K) \), \( \omega_1 \in B(H) \ast \) and \( \omega_2 \in B(K) \ast \).

**Lemma 5.6.3.** For any \( \omega_1 \in B(H) \ast \), \( \omega_1 \otimes id \) has a unique weak* continuous linear extension \( R(\omega_1) : B(H \otimes_2 K) \rightarrow B(K) \) with \( \| R(\omega_1) \|_{cb} \leq \| \omega_1 \| \). Similarly, for any \( \omega_2 \in B(K) \ast \), \( id \otimes \omega_2 \) has a unique weak* continuous extension \( L(\omega_2) : B(H \otimes_2 K) \rightarrow B(H) \) with \( \| L(\omega_2) \|_{cb} \leq \| \omega_2 \| \).

**Proof.** Let \( \omega_1 \in B(H) \ast \). For \( u \in B(H \otimes_2 K) \), we define \( R(\omega_1)(u) \in B(K) \) by letting \( (R(\omega_1)(u))(\omega_2) = \langle u, \omega_1 \otimes \omega_2 \rangle \) for all \( \omega_2 \in B(K) \ast \). Note that \( \omega_1 \otimes \omega_2 \in B(H) \ast \otimes B(K) \ast \subseteq B(H \otimes_2 K) \ast \). So the above definition makes sense. Clearly, \( R(\omega_1) : B(H \otimes_2 K) \rightarrow B(K) \) is a linear extension of \( \omega_1 \otimes id \). From Theorem 5.6.2 and Corollary 5.3.3, we have the identifications

\[
B(H \otimes_2 K) \cong (B(H) \hat{\otimes} B(K) \ast) \ast \text{ and } \lambda : B(H \otimes_2 K) \cong CB(B(H) \ast, B(K)),
\]

where the complete isometry \( \lambda \) is given by \( (\lambda(u)(\omega_1))(\omega_2) = \langle u, \omega_1 \otimes \omega_2 \rangle \) for all \( \omega_1 \in B(H) \ast \) and \( \omega_2 \in B(K) \ast \). Note that

\[
(R(\omega_1)(u))(\omega_2) = \langle u, \omega_1 \otimes \omega_2 \rangle = (\lambda(u)(\omega_1))(\omega_2)
\]

for all \( \omega_2 \in B(K) \ast \). Thus \( R(\omega_1)(u) = \lambda(u)(\omega_1) \) for all \( u \in B(H \otimes_2 K) \). If \( u \in M_n(B(H \otimes_2 K)) \) \( (n \in \mathbb{N}) \), then

\[
R(\omega_1)^{(n)}(u) = [R(\omega_1)(u_{i,j})] = [\lambda(u_{i,j})(\omega_1)] = \lambda^{(n)}(u)(\omega_1),
\]

and \( \| R(\omega_1)^{(n)}(u) \| \leq \| \lambda^{(n)}(u) \| \| \omega_1 \| = \| u \| \| \omega_1 \| \). So \( \| R(\omega_1)^{(n)} \| \leq \| \omega_1 \| \) for all \( n \in \mathbb{N} \). It follows that \( \| R(\omega_1) \|_{cb} \leq \| \omega_1 \| \).
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Since \( \omega_1 \otimes \omega_2 \) is weak* continuous on \( B(H \otimes_2 K) \) for all \( \omega_2 \in B(K)_* \), it follows from the definition of \( R(\omega_1) \) that \( R(\omega_1) \) is weak*-continuous. Since \( B(H) \otimes B(K) \) is weak* dense in \( B(H \otimes_2 K) \), the extension is unique. \( \square \)

**Definition 5.6.4.** Let \( H \) and \( K \) be Hilbert spaces, \( \omega_1 \in B(H)_* \), and \( \omega_2 \in B(K)_* \). Let \( R(\omega_1) \) and \( L(\omega_2) \) be the weak* continuous extension of \( \omega_1 \otimes \text{id} \) and \( \text{id} \otimes \omega_2 \) on \( B(H \otimes_2 K) \), respectively. Then \( R(\omega_1) \) is called the right slice mapping determined by \( \omega_1 \), and \( L(\omega_2) \) is called the left slice mapping determined by \( \omega_2 \). In the following, we will still write \( \omega_1 \otimes \text{id} \) for \( R(\omega_1) \) and \( \text{id} \otimes \omega_2 \) for \( L(\omega_2) \).

**Definition 5.6.5.** Let \( H \) be a Hilbert space, and \( W \) be an operator space which is the dual of an operator space \( V \). We say that a mapping \( \varphi : W \to B(H) \) is a dual realization of \( W \) on \( H \) if it is a weak* homeomorphic completely isometric injection.

**Proposition 5.6.6.** Let \( H, V, W, \) and \( \varphi \) be the same as above. Then \( \varphi \) is a complete weak* homeomorphism in the sense that each \( \varphi^{(n)} : M_n(W) \to M_n(B(H)) \) is weak* homeomorphic, where \( M_n(W) \) and \( M_n(B(H)) \) are dual operator spaces via the identifications \( M_n(W) = M_n(V^*) \cong T_n(V)^* \) and \( M_n(B(H)) = M_n(B(H)_*) \cong (T_n(B(H)_*))^* \).

**Proof.** Let \( T^o = (T_{ij}^o) \) be a net in \( M_n(W) \). Then \( T^o \to T = (T_{ij}) \in M_n(W) \) in the weak* topology if and only if \( |T^o(v) - T(v)| \to 0 \) for all \( v = (v_{ij}) \in T_n(V) \) if and only if \( |T_{ij}^o(v_{ij}) - T_{ij}(v_{ij})| \to 0 \) for all \( 1 \leq i, j \leq n \) and all \( v_{ij} \in V \) if and only if \( T_{ij}^o \to T_{ij} \) in the weak* topology for all \( 1 \leq i, j \leq n \) if and only if \( \varphi(T_{ij}^o) \to \varphi(T_{ij}) \) in the weak* topology for each \( 1 \leq i, j \leq n \) (since \( \varphi \) is weak* homeomorphic) if and only if \( |\varphi(T_{ij}^o)(u) - \varphi(T_{ij})(u)| \to 0 \) for all \( u \in B(H) \), if and only if \( |\varphi^{(n)}(T^o)(U) - \varphi^{(n)}(T)(U)| \to 0 \) for all \( U \in M_n(B(H)_*) \) if and only if \( \varphi^{(n)}(T^o) \to \varphi^{(n)}(T) \) in the weak* topology if and only if \( \varphi^{(n)} \) is weak* homeomorphic for each \( n \in \mathbb{N} \). \( \square \)

The following theorem is cited from [13, Proposition 3.2.4]. It tells us that the dual space of any complete operator space has a dual realization.

**Theorem 5.6.7.** If \( V \) is a complete operator space, then \( V^* \) has a dual realization on a Hilbert space \( H \).
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In the following, $V^*$ is always identified with its image in $B(H)$ under its dual realization. In particular, $V^*$ is a weak* closed subspace of $B(H)$.

**Definition 5.6.8.** Let $V^*$ and $W^*$ be the dual operator spaces of complete operator spaces $V$ and $W$, respectively, with dual realizations $\pi_1 : V^* \to B(H)$ and $\pi_2 : W^* \to B(K)$. We have two dual tensor products associated with these representations. On one hand, we define the normal spatial tensor product $V^\otimes W^*$ to be the weak* closure of $V^* \otimes W^*$ in $B(H \otimes_2 K)$, where $B(H \otimes_2 K) = (B(H)_d \odot B(K)_d)^*$.

On the other hand, we define the normal Fubini tensor product $V^\otimes_F W^*$ by

$$V^\otimes_F W^* = \{ u \in B(H \otimes_2 K) : (\omega_1 \otimes \text{id})(u) \in W^* \text{ and } (\text{id} \otimes \omega_2)(u) \in V^* \text{ for all } \omega_1 \in B(H)_d, \omega_2 \in B(K)_d \}.$$ 

**Theorem 5.6.9.** Let $V$ and $W$ be complete operator spaces. Then

$$V^\otimes W^* \subseteq V^\otimes_F W^*.$$ 

**Proof.** Let $u \in V^\otimes W^*$. We want to show that $u \in V^\otimes_F W^*$. That is, we want to show that $(\omega_1 \otimes \text{id})(u) \in W^*$ and $(\text{id} \otimes \omega_2)(u) \in V^*$ for all $\omega_1 \in B(H)_d$ and $\omega_2 \in B(K)_d$. By definition, if $u \in V^* \otimes W^*$, then $(\omega_1 \otimes \text{id})(u)$ in $W^*$ and $(\text{id} \otimes \omega_2)(u) \in V^*$. In general, there is a net $u_\lambda \in V^* \otimes W^*$ converging to $u$ in the weak* topology on $B(H \otimes_2 K)$. Since $\omega_1 \otimes \text{id} : B(H \otimes_2 K) \to B(K)$ is weak*-weak* continuous, $(\omega_1 \otimes \text{id})(u_\lambda)$ converges to $(\omega_1 \otimes \text{id})(u)$ in the weak* topology on $B(K)$. Therefore, $(\omega_1 \otimes \text{id})(u) \in W^*$, since $W^*$ is weak* closed in $B(K)$. Similarly, $(\text{id} \otimes \omega_2)(u) \in V^*$.

**Definition 5.6.10.** A C*-algebra $M$ is called a $W^*$-algebra if it is a dual space of a Banach space.

In general, given complete operator spaces $V$ and $W$, the normal spatial tensor product $V^* \otimes W^*$ may not be equal to the normal Fubini tensor product $V^* \otimes_F W^*$. We will see in the following that it is the case when both $V^*$ and $W^*$ are $W^*$-algebras (see [2, page 40]).

**Definition 5.6.11.** We use $V^\otimes W^*$ to denote the weak* closure of $V^* \otimes W^*$ in $(V \otimes W)^*$, called the abstract normal spatial tensor product of $V^*$ and $W^*$.
Given dual realizations \( \pi_1 \) and \( \pi_2 \) as above, we have from Corollary 5.5.7 that \( \pi_1 \) and \( \pi_2 \) are the adjoints of complete quotient mappings

\[
\pi_1^* : B(H)_* \longrightarrow V \quad \text{and} \quad \pi_2^* : B(K)_* \longrightarrow W.
\]

It follows from Theorem 5.4.2 that

\[
\pi_* = \pi_1^* \otimes \pi_2^* : B(H)_* \otimes B(K)_* = B(H \otimes_2 K)_* \longrightarrow V \bar{\otimes} W
\]

is a complete quotient mapping, and in turn,

\[
\pi = (\pi_*)^* : (V \bar{\otimes} W)^* \longrightarrow B(H \otimes_2 K)
\]

is a dual realization of \((V \bar{\otimes} W)^*\) (see Corollary 5.5.7 again).

**Theorem 5.6.12.** Let \( V \) and \( W \) be complete operator spaces. \( \pi_1 \) and \( \pi_2 \) are dual realizations of \( V^* \) and \( W^* \), respectively. \( \pi \) is defined as above. Then \( \pi \) is a weak* homeomorphic completely isometric mapping of \((V \bar{\otimes} W)^*\) onto \( V^* \bar{\otimes}_F W^* \), and it carries \( V^* \bar{\otimes}_F W^* \) onto \( V^* \bar{\otimes}_F W^* \).

**Proof.** From the argument above, we know that \( \pi = (\pi_*)^* \) is a complete quotient mapping. Also, \( \pi \) is a dual realization of \((V \bar{\otimes} W)^*\) on \( H \otimes_2 K \). So, \( \pi((V \bar{\otimes} W)^*) \) is weak* closed in \( B(H \otimes_2 K) \). It is equivalent to say that \( \text{ran} \pi_* \) is norm closed in \( V \bar{\otimes} W \) (see [5, Theorem VI.1.10]). Thus, \( \pi((V \bar{\otimes} W)^*) = (\ker \pi_*)^\perp \) (see [7, Theorem VI.6.2]). Combining this result with Theorem 5.4.2, we have

\[
\pi((V \bar{\otimes} W)^*) = (B(H)_* \otimes \ker \pi_2 + \ker \pi_1 \otimes B(K)_*)^\perp.
\]

It follows that, for any \( u \in B(H \otimes_2 K) \), \( u \in \pi((V \bar{\otimes} W)^*) \) if and only if \( \omega_1 \otimes \omega_2(u) = 0 \) when either \( \omega_1 \in \ker \pi_1 \) or \( \omega_2 \in \ker \pi_2 \). Note that \( \omega_1((id \otimes \omega_2)(u)) = \omega_2((\omega_1 \otimes id)(u)) = \omega_1 \otimes \omega_2(u) \). Thus, \( u \in \pi((V \bar{\otimes} W)^*) \) if and only if \( \omega_1 \otimes id) \) is \( (\ker \pi_2)^\perp \) and \( (id \otimes \omega_2)(u) \in (\ker \pi_1)^\perp \) for all \( \omega_1 \in B(H)_* \) and \( \omega_2 \in B(K)_* \). Similarly, we have \( \pi_1(V^*) = (\ker \pi_1)^\perp \) and \( \pi_2(W^*) = (\ker \pi_2)^\perp \). So, \( u \in \pi((V \bar{\otimes} W)^*) \) if and only if \( \omega_1 \otimes id)(u) \in W^* \) and \( (id \otimes \omega_2)(u) \in V^* \) for all \( \omega_1 \in B(H)_* \) and \( \omega_2 \in B(K)_* \), i.e., \( u \in V^* \bar{\otimes}_F W^* \). This shows that \( \pi \) maps \((V \bar{\otimes} W)^*\) onto \( V^* \bar{\otimes}_F W^* \).

The last statement of the theorem is true since \( \pi \) is a weak* homeomorphism, and hence

\[
\pi(V^* \bar{\otimes} W^*) = \pi((V^* \bar{\otimes} W^*)^*) = V^* \bar{\otimes} W^*.
\]
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Here, we note that when $V^* \otimes W^*$ is considered as a subspace of $(V \otimes W)^*$, $\pi(V^* \otimes W^*)$ is equal to the image of the natural embedding of the algebraic tensor product $V^* \otimes W^*$ into $B(H \otimes_2 K)$. □

Remark 5.6.13. Due to the above theorem, we will generally use the notation $V^* \otimes W^*$ for the abstract normal spatial tensor product, and call $(V \otimes W)^*$ the abstract normal Fubini tensor product of $V^*$ and $W^*$.

5.7. Extension of Fubini Theorem

The main purpose of this section is to extend the aforesaid Fubini Theorem on trace class operators to the preduals of general von Neumann algebras.

Definition 5.7.1. A von Neumann algebra $\mathcal{R}$ on a Hilbert space $H$ is a unital $*$-subalgebra on $H$, which is closed in the weak operator topology.

The following theorem can be found in [20, VI.7.1], which gives a few characterizations of von Neumann algebras.

Theorem 5.7.2. Let $M$ be a $*$-algebra of $B(H)$ containing the unit of $B(H)$. Then the following assertions are equivalent.

(a) $M = M''$ (the double commutant of $M$).

(b) $M$ is closed in $B(H)$ in the weak operator topology.

(c) $M$ is closed in $B(H)$ in the strong operator topology.

(d) $M$ is closed in $B(H)$ in the weak* topology.

In this case, $M$ is a $W^*$-algebra.

Conversely, we have

Theorem 5.7.3. (Sakai Theorem) A C*-algebra $\mathcal{R}$ is $*$-isomorphic to a von Neumann algebra if and only if it is a $W^*$-algebra. If it is the case, the $*$-isomorphism is also a weak*-homeomorphism and the predual of $\mathcal{R}$ is essentially unique.

Examples

(1) For any Hilbert space $H$, $B(H)$ is a von Neumann algebra with predual $TC(H)$, the trace class operators on $H$.

(2) Consider a $\sigma$-finite measure space $(X, M, \mu)$. Two measurable functions on $X$ are said to be equivalent if they are equal $\mu$-almost everywhere. Let $L^1(X)$ be

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the Banach space of all those equivalent classes of integrable functions on $X$, $L^2(X)$ the Hilbert space of all those equivalent classes of square integrable functions on $X$, and $L^\infty(X)$ the $*$-algebra of essentially bounded measurable functions on $X$, where $'*$' takes the conjugate operation, i.e., $f^*(x) = \overline{f(x)}$ for all $f \in L^\infty(X)$ and all $x \in X$. Define $\pi : L^\infty(X) \to B(L^2(X))$ by $\pi(f)(g) = fg$ for all $f \in L^\infty(X)$ and $g \in L^2(X)$. It is easy to see that $\pi$ is a $*$-homomorphism. Moreover, if $\pi(f) = 0$ for some $f \in L^\infty(X)$, then for any $g \in L^2(X)$, $fg = 0$ almost everywhere. Since $(X, M, \mu)$ is $\sigma$-finite, we can suppose that $X = \bigcup_{n=1}^{\infty} \Delta_n$, where each $\Delta_n$ is measurable with finite measure. Then the characteristic function $\chi_{\Delta_n}$ is in $L^2(X)$ for all $n \in \mathbb{N}$.

So, for all $n \in \mathbb{N}$, $f \chi_{\Delta_n} = 0$ almost everywhere, and so is $f$. This shows that $\pi$ is injective. Thus, $\pi$ is a $*$-isomorphism, and $L^\infty(X)$ is a $*$-subalgebra of $C^*$-algebra $B(L^2(X))$. It follows that $L^\infty(X)$ is also a $C^*$-algebra. Since $L^\infty(X) = L^1(X)^*$, $L^\infty(X)$ is a $W^*$-algebra. By Sakai Theorem, $L^\infty(X)$ is a von Neumann algebra.

(3) Suppose that $G$ is a locally compact group and $\lambda$ is a fixed left Haar measure (for more basic concepts of locally compact group, see [15]). If $A \subseteq G$ and $\lambda(A) = 0$, then $A$ is said to be a $\lambda$-null set. If $A \cap F$ is a $\lambda$-null set for every compact subset $F$ of $X$, then $A$ is said to be a locally $\lambda$-null set. Two functions are said to be equal locally $\lambda$-almost everywhere if they are equal except for a locally $\lambda$-null set. A function $f$ is said to be bounded locally $\lambda$-almost everywhere if it is bounded except for a locally $\lambda$-null set. Two bounded locally $\lambda$-almost everywhere functions are said to be essentially equivalent if they are equal except for a locally $\lambda$-null set.

Let $L_p(G)$ ($1 \leq p < \infty$) be the Banach space of all those equivalent classes of $\lambda$-measurable functions on $G$ such that

$$||f||_p := \left( \int_G |f(x)|^p dx \right)^{1/p} < \infty.$$  

Let $L_\infty(G)$ be the $*$-algebra of all the equivalent classes of functions which are bounded locally almost everywhere with $||f||_\infty = \inf_E \sup_{x \in E} |f(x)|$, where the infimum is taken over all those locally $\lambda$-null set $E$ of $G$. Define $\pi : L_\infty(G) \to B(L_2(G))$ by $\pi(f)(g) = fg$ for all $f \in L_\infty(G)$ and $g \in L_2(G)$. Then it is easy to see that $\pi$ is a $*$-homomorphism. Moreover, if $\pi(f) = 0$ for some $f \in L_\infty(G)$, then for any $g \in L_2(G)$, $fg = 0$ almost everywhere. For any compact subset $K \subseteq G$, $\chi_K \in L_2(G)$. So, $f \chi_K = 0$ almost everywhere, i.e., $f = 0$ locally almost everywhere. This shows

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that $\pi$ is a $*$-isomorphism. For the similar reason to Example (2), we see that $L_\infty(G)$ is a von Neumann algebra.

Every von Neumann algebra $\mathcal{R} \subseteq B(H)$ has its natural operator space structure, which induces an operator space structure on $\mathcal{R}^*$ and hence on $\mathcal{R}_*$. It is known that $\mathcal{R} \cong (\mathcal{R}_*)^*$ as operator spaces and the inclusion mapping $\mathcal{R} \to B(H)$ is a dual realization. Therefore, by Theorem 5.7.3, we have

**Theorem 5.7.4.** Every von Neumann algebra and hence every $W^*$-algebra has a dual realization.

**Definition 5.7.5.** Let $M$ and $N$ be von Neumann algebras acting on Hilbert spaces $H$ and $K$, respectively. The von Neumann algebra tensor product of $M$ and $N$, denoted by $M \otimes N$, is the closure of the algebraic tensor product $M \otimes N$ in the weak operator topology on $B(H \otimes_2 K)$.

Since the inclusions $M \hookrightarrow B(H)$ and $N \hookrightarrow B(K)$ are dual realizations, we have

**Theorem 5.7.6.** For von Neumann algebras $\mathcal{R}$ and $\mathcal{S}$, the normal spatial tensor product of $\mathcal{R}$ and $\mathcal{S}$ is just the von Neumann algebra tensor product $\mathcal{R} \otimes \mathcal{S}$.

The following theorem is one of the fundamental results on von Neumann algebras.

**Theorem 5.7.7.** If $\mathcal{R} \subseteq B(H)$ and $\mathcal{S} \subseteq B(K)$ are von Neumann algebras, then $(\mathcal{R} \otimes \mathcal{S})' = \mathcal{R}_* \otimes \mathcal{S}_*$, where the three primes denote the commutants in $B(H \otimes_2 K)$, $B(H)$ and $B(K)$, respectively.

From Definition 5.7.5 and Theorem 5.7.6, we see that $B(H \otimes_2 K) = B(H) \bar{\otimes} B(K)$. Then we can rewrite the aforesaid Fubini theorem (see Theorem 5.6.2) as

$$(B(H) \bar{\otimes} B(K))_* \cong B(H)_* \bar{\otimes} B(K)_*.$$ 

Thus, The following theorem is an extension of the aforesaid Fubini theorem. We call this theorem the Fubini theorem for von Neumann algebras.

**Theorem 5.7.8.** Given von Neumann algebras $\mathcal{R} \subseteq B(H)$ and $\mathcal{S} \subseteq B(K)$, we have a natural complete isometry $(\mathcal{R} \bar{\otimes} \mathcal{S})_* = \mathcal{R}_* \bar{\otimes} \mathcal{S}_*$, and hence

$$\mathcal{R} \bar{\otimes} \mathcal{S} \cong (\mathcal{R}_* \bar{\otimes} \mathcal{S}_*)^* \cong \mathcal{R}_F \mathcal{S}.$$
PROOF. By Theorem 5.7.3 and Theorem 5.6.12, it suffices to show that \( \mathcal{R}\mathcal{S} = \mathcal{R}_F\mathcal{S} \). Due to Theorem 5.6.9, we only need to show that \( \mathcal{R}\mathcal{S} \subseteq \mathcal{R}_F\mathcal{S} \).

Let \( u \in \mathcal{R}_F\mathcal{S} \). Since \( \mathcal{R}\mathcal{S} \) is a von Neumann algebra, by Theorem 5.7.2, it suffices to show that \( u \) commutes with all operators in the commutant \( (\mathcal{R}\mathcal{S})' \). By Theorem 5.7.7, the latter is just \( \mathcal{R}\mathcal{S}' \), and thus it suffices to show that \( ux = xu \) for all \( x \in \mathcal{R}\mathcal{S}' \). Since the von Neumann tensor product \( \mathcal{R}\mathcal{S}' \) is also equal to the closure of its algebraic tensor product \( \mathcal{R}'\mathcal{S}' \) in the strong operator topology, it suffices to show that \( u(r' \otimes s') = (r' \otimes s')u \), or equivalently, \( u(r' \otimes I) = (r' \otimes I)u \) for all \( r' \in \mathcal{R}' \) and \( u(I \otimes s') = (I \otimes s')u \) for all \( s' \in \mathcal{S}' \). Turning to the latter, by the identification \( B(H \otimes K) \cong CB(B(H)_*, B(K)) \), each \( x \in B(H \otimes K) \) can be viewed as an element in \( CB(B(H)_*, B(K)) \), which maps \( \omega_1 \) to \( (\omega_1 \otimes id)(x) \). Thus, in order to show that \( u(I \otimes s') = (I \otimes s')u \), it suffices to show that \( (u(I \otimes s'))(\omega_1) = (((I \otimes s')u)(\omega_1) \), i.e., \( (\omega_1 \otimes id)(u(I \otimes s')) = (\omega_1 \otimes id)((I \otimes s')u) \) for all \( \omega_1 \in \mathcal{R}_* \).

By the assumption that \( u \in \mathcal{R}_F\mathcal{S} \), \( (\omega_1 \otimes id)(u) \in \mathcal{S} \) for all \( \omega_1 \in \mathcal{R}_* \). So, for all \( s' \in \mathcal{S}' \) and \( \omega_1 \in \mathcal{R}_* \),

\[
((\omega_1 \otimes id)(u))s' = s'((\omega_1 \otimes id)(u)).
\]

Note that for all \( \omega_2 \in B(K)_* \),

\[
(((\omega_1 \otimes id)(u))s', \omega_2) = ((\omega_1 \otimes id)(u), s'\omega_2) = (u, \omega_1 \otimes (s'\omega_2)) = (u(I \otimes s'), \omega_1 \otimes \omega_2) = ((\omega_1 \otimes id)(u(I \otimes s')), \omega_2),
\]

i.e., \( ((\omega_1 \otimes id)(u))s' = (\omega_1 \otimes id)(u(I \otimes s')) \). Similarly, \( s'((\omega_1 \otimes id)(u)) = (\omega_1 \otimes id)((I \otimes s')u) \). Therefore, \( (\omega_1 \otimes id)(u(I \otimes s')) = (\omega_1 \otimes id)((I \otimes s')u) \) for all \( \omega_1 \in \mathcal{R}_* \) and \( s' \in \mathcal{S}' \), i.e., \( u(I \otimes s') = (I \otimes s')u \) for all \( s' \in \mathcal{S}' \).

By the same argument, we have \( u(r' \otimes I) = (r' \otimes I)u \) for all \( r' \in \mathcal{R}' \). Therefore, \( u \in \mathcal{R}\mathcal{S} \). The proof is complete.  \( \square \)
CHAPTER 6

Injective Tensor Products

As we know, for vector spaces $X$ and $Y$, there is an algebraic isomorphism between $BL(X \times Y)$ and $L(X \otimes Y, \mathbb{C})$ (see Section 2.6). In this chapter, we will discuss the dual spaces of injective tensor products $V \otimes^\vee W$ of operator spaces $V$ and $W$. The materials presented in this chapter are mainly cited from [12] and [19].

We still start with the case of Banach spaces. We first review that, for Banach spaces $X$ and $Y$, the dual space $(X \otimes^\delta Y)^*$ is isometrically isomorphic to a subspace of $BL(X \times Y)$ with finite integral norm, i.e., $B_I(X \times Y)$ (see [19]). This is proved in detail in Section 6.2. Let $I(X,Y)$ denote the normed space of all integral maps from Banach space $X$ to Banach space $Y$. Then $B_I(X \times Y) \cong I(X,Y^*)$ is an isometric identification via $F \mapsto T_F$ defined by $(T_F x, y) = F(x,y)$ for all $x \in X, y \in Y$, and $F \in B_I(X \times Y)$. So $(X \otimes^\delta Y)^* \cong I(X,Y^*)$ is also an isometry. Following this idea, let $I(V, W)$ be the space of all integral maps from operator space $V$ to operator space $W$. We will see that the corresponding isometry $(V \otimes^\vee W)^* \cong I(V, W^*)$ does not always hold. However, we present some equivalent conditions for which the isometry holds.

6.1. Introduction to Injective Tensor Products

In this section, we first have a review on Banach space injective tensor product. Then we present the definition of operator space injective tensor product and its fundamental properties (see [12]).

**Definition 6.1.1.** Let $E$ and $F$ be Banach spaces. The Banach space injective tensor product norm of $u \in E \otimes F$ is defined by

$$
\|u\|_\lambda = \sup \{|(f \otimes g)(u)| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\}.
$$

(35)

We let $E \otimes_\lambda F = (E \otimes F, \| \cdot \|_\lambda)$, and denote the completion of $E \otimes_\lambda F$ by $E \otimes^\delta F$, called the Banach space injective tensor product of $E$ and $F$.  

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THEOREM 6.1.2. The Banach space injective tensor product is also determined by the natural injection \( \theta : E \otimes F \rightarrow B(E^*, F) \) defined by
\[
\theta(x \otimes y)(f) = f(x)y
\] (36)
for all \( x \in E, y \in F \) and \( f \in E^* \). Namely, for all \( u \in E \otimes F \), we have \( \|u\|_A = \|\theta(u)\|_A \).

Turning to the case of operator spaces, the operator space injective tensor product is defined in a similar way. We will see that it has the property of “injectivity” (see Theorem 6.1.13).

DEFINITION 6.1.3. Let \( V \) and \( W \) be operator spaces. The injective matrix norm \( \| \cdot \|_V \) on \( V \otimes W \) is defined by
\[
\|u\|_V = \sup \{ \| (f \otimes g)^{(n)}(u) \| : f \in M_p(V^*), g \in M_q(W^*), \|f\|, \|g\| \leq 1, p, q \in \mathbb{N} \} \tag{37}
\]
for all \( u \in M_n(V \otimes W) \) (\( n \in \mathbb{N} \)).

We show in the following that \( \| \cdot \|_V \) is an operator space matrix norm on \( V \otimes W \).

THEOREM 6.1.4. Let \( V \) and \( W \) be operator spaces. Then the injective matrix norm is determined by the natural embedding \( \theta : V \otimes W \rightarrow CB(V^*, W) \) defined by
\[
\theta(v \otimes w)(f) = f(v)w \tag{38}
\]
for all \( f \in V^*, v \in V \) and \( w \in W \). Namely, for all \( n \in \mathbb{N} \) and \( u \in M_n(V \otimes W) \), we have \( \|u\|_V = \|\theta(u)\|_{M_n(CB(V^*, W))} \). So, \( \| \cdot \|_V \) is an operator space matrix norm.

PROOF. For any \( f \in V^*, g \in W^*, v \in V \), and \( w \in W \), we have
\[
g(\theta(v \otimes w)(f)) = g(f(v)w) = f(v)g(w) = (f \otimes g)(v \otimes w).
\]
By linearity, \( g(\theta(u)(f)) = (f \otimes g)(u) \) for all \( u \in V \otimes W \). If \( u \in M_n(V \otimes W) \), then \( \theta^{(n)}(u) \in M_n(CB(V^*, W)) = CB(V^*, M_n(W)) \). Thus, for \( f \in M_n(V^*) \),
\[
\theta^{(n)}(u)^{(p)}(f) \in M_p(M_n(W)) = M_{pm}(W).
\]
If \( g \in M_q(W^*) = CB(W, M_q) \), then
\[
g^{(pm)}(\theta^{(n)}(u)^{(p)}(f)) = g^{(pm)}([\theta(u)(f_{i,j})]) = [g(\theta(u)(f_{i,j}))]_{pm}
\]
For any element $v \in M_n(V)$, by Lemma 4.3.2, there exists a complete contraction
\[ \varphi : V \to M_n \] such that $\|\varphi^{(n)}(v)\| = \|v\|$. Meanwhile, for any complete contraction
\[ g : V \to M_q \] with $q \in \mathbb{N}$ arbitrary, we have $\|g^{(n)}(v)\| \leq \|g\| \|v\| \leq \|v\|$. Therefore,
\[ \|v\| = \sup\{\|g^{(n)}(v)\| : g \in M_q(V^*), \|g\| \leq 1, q \in \mathbb{N}\}. \tag{39} \]
Replacing $v$ by $\theta^{(n)}(u) (f)$ in $M_{pn}(W)$, we have
\[ \|\theta^{(n)}(u) (f)\| = \sup\{\|g^{(m)}(\theta^{(n)}(u) (f))\| : g \in M_q(W^*), \|g\| \leq 1, q \in \mathbb{N}\}. \]
Then
\[ \|\theta^{(n)}(u)\|_\Theta = \sup\{\|\theta^{(n)}(u) (f)\| : f \in M_p(V^*), \|f\| \leq 1, p \in \mathbb{N}\} \]
\[ = \sup\{\|g^{(m)}(\theta^{(n)}(u) (f))\| : f \in M_p(V^*), g \in M_q(W^*), \|f\|, \|g\| \leq 1, p, q \in \mathbb{N}\} \]
\[ = \sup\{\|(f \otimes g)^{(n)}(u)\| : f \in M_p(V^*), g \in M_q(W^*), \|f\|, \|g\| \leq 1, p, q \in \mathbb{N}\} \]
\[ = \|u\|_\Theta. \]
Therefore, $\theta$ is a completely isometric injection. \qed

**Remark 6.1.5.** In fact, by [9], we have the following complete isometry
\[ V \hat{\otimes} W \cong CF(V^*, W) \]
where $CF(V^*, W)$ denotes the closure of all finite rank maps in $CB(V^*, W)$. Thus, in case that $V$ or $W$ is finite-dimensional, we have complete isometry
\[ V \hat{\otimes} W \cong CB(V^*, W). \]

We denote the operator space $(V \otimes \nu W, \|\cdot\|_\nu)$ by $V \hat{\otimes} \nu W$, and denote the completion of $V \otimes \nu W$ by $V \hat{\otimes} W$, called the **operator space injective tensor product** of $V$ and $W$. By Theorem 6.1.4, we have a completely isometric injection: $V \otimes \nu W \hookrightarrow CB(V^*, W)$. In case that $W$ is complete (and hence $CB(V^*, W)$ is complete), the completely isometric injection is extended to $V \hat{\otimes} W \hookrightarrow CB(V^*, W)$. 

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COROLLARY 6.1.6. Let $V$ and $W$ be operator spaces. Then the identity mapping on $V \otimes W$ induces a contractive injective mapping $V \otimes W \rightarrow V \otimes^l W$.

**Proof.** By Theorem 6.1.2, $V \otimes^l W$ can be identified with a subspace of $B(V^*, W)$. Meanwhile, $B(V^*, W) \subseteq B(V^{**}, W^{**})$ as normed spaces. So $V \otimes^l W$ can be identified with a subspace of $B(V^*, W^{**})$. Since the latter is complete, we have the isometric injection $V \otimes^l W \rightarrow B(V^*, W^{**})$. Similarly, by Theorem 6.1.4, we have the operator space embedding $V \otimes_v W \rightarrow CB(V^*, W)$. Together with the natural embedding $CB(V^*, W) \rightarrow CB(V^*, W^{**})$, we get the operator space embedding $V \otimes_v W \rightarrow CB(V^*, W^{**})$. It follows from Proposition 4.5.5 that $V \otimes W \rightarrow CB(V^*, W^{**})$ is an operator space embedding. Now, look at the following diagram

$$
\begin{array}{ccc}
V \otimes W & \rightarrow & CB(V^*, W^{**}) \\
\downarrow & & \downarrow \\
V \otimes^l W & \rightarrow & B(V^*, W^{**})
\end{array}
$$

where the row mappings are isometric and the right column mapping is a contractive injection. From Equality (36) and Equality (38), we see that the above diagram is commutative. So, the left column mapping must be contractive and injective. □

We denote by $CB^*(V^*, W^*)$ the subspace of $CB(V^*, W^*)$ consisting of all weak* continuous mappings from $V^*$ into $W^*$. Then $CB^*(V^*, W^*)$ is an operator subspace of $CB(V^*, W^*)$. We have

**Theorem 6.1.7.** Let $V$ be an operator space and let $i_V : V \hookrightarrow V^{**}$ be the canonical embedding. Then, for all $n \in \mathbb{N}$,

$$CB^*(V^*, M_n) = (i_V)^{(n)}(M_n(V)).$$

Equivalently, $CB^*(V^*, M_n) \cong M_n(V)$ is a complete isometry.

**Proof.** It is easy to see that $F = [F_{i,j}] : V^* \rightarrow M_n$ is weak* continuous if and only if each $F_{i,j}$ is weak* continuous. The latter one implies that $F_{i,j} = i_V(v_{i,j})$ for some $v_{i,j} \in V$. Thus, $F \in CB^*(V^*, M_n)$ if and only if $F_{k,l} \in i_V(V)$ for all $1 \leq k, l \leq n$, i.e., $F \in CB^*(V^*, M_n)$ if and only if $F = [F_{k,l}] \in i_V^{(n)}(M_n(V))$. Thus

$$CB^*(V^*, M_n) = i_V^{(n)}(M_n(V))$$

as normed spaces. Since the canonical inclusion $i_V : V \hookrightarrow V^{**}$ is a completely isometric injection (see [13, Proposition 3.2.1]), equality (40) is thus an operator
space identification. This can also be easily seen from the following commutative diagram.

\[
\begin{array}{ccc}
M_n(V) & \longrightarrow & M_n(V^{**}) \\
\downarrow & & \downarrow \\
CB^*(V^*, M_n) & \longrightarrow & CB(V^*, M_n)
\end{array}
\]

For all \( n \in \mathbb{N} \), both rows are operator space embeddings, and the right column is a complete isometric identification. So, the left column is also completely isometric. □

The following lemma can be found in [13, Proposition 4.2.5].

**Lemma 6.1.8.** Given any complete contraction \( F : V^* \longrightarrow M_n \), there is a net of weak* continuous complete contractions \( F_\gamma : V^* \longrightarrow M_n \), which converges to \( F \) in the point-norm topology. For each \( \gamma \), there is a unique element \( v_\gamma \in M_n(V) \) for which \( F_\gamma(f) = f^{(n)}(v_\gamma) \) for all \( f \in V^* \).

**Theorem 6.1.9.** Given operator spaces \( V \) and \( W \), the natural embedding

\[ \theta : V^* \otimes V \hookrightarrow CB(V, W) \]

is completely isometric.

**Proof.** If \( \Theta : V^* \otimes V \hookrightarrow CB(V^{**}, W) \) denotes the natural embedding as in Theorem 6.1.4, then \( \theta(u) = \Theta(u)|_V \), where \( V \) is regarded as a subspace of \( V^{**} \). By the proof of Theorem 6.1.4, we see that for \( g \in W^* \) and \( u \in V^* \otimes W \), \( g(\theta(u)(v)) = (v \otimes g)(u) \) for all \( v \in V \). Also, if \( u \in M_p(V^* \otimes W), v \in M_p(V) \), and \( g \in M_q(W^*) \), then \( g^{(p)}(\theta^{(n)}(u)(p))(v) = (v \otimes g)^{(n)}(u) \).

Follows the same argument as in the proof of Theorem 6.1.4, we have

\[
\|\theta^{(n)}(u)\|_{cb} = \sup\{\|(\theta^{(n)}(u))^{(p)}\| : p \in \mathbb{N}\}
\]

\[
= \sup\{\|(\theta^{(n)}(u)^{\otimes p})(v)\| : v \in M_p(V), \|v\| \leq 1, p \in \mathbb{N}\}
\]

\[
= \sup\{\|(\theta^{(n)}(u)(p))(v)\| : v \in M_p(V)\|_{\leq 1}, g \in M_q(W^*)\|_{\leq 1}, p, q \in \mathbb{N}\}
\]

\[
= \sup\{\|(v \otimes g)^{(n)}(u)\| : v \in M_p(V)\|_{\leq 1}, g \in M_q(W^*)\|_{\leq 1}, p, q \in \mathbb{N}\}.
\]

Given \( F \in M_p(V^{**}) = CB(V^*, M_p) \) with \( \|F\| \leq 1 \). By Lemma 6.1.8, we can find a net \( (v_\gamma) \) in \( M_p(V) = CB^*(V^*, M_p) \) with \( \|v_\gamma\|_{cb} \leq 1 \), such that \( f^{(p)}(v_\gamma) \longrightarrow F(f) \) for
all $f \in V^*$. With this argument, we have

$$\sup\{\|f(u \otimes g)^{(n)}(u)\| : u \in M_p(V), g \in M_q(W^*)\|_{\leq 1}, p, q \in \mathbb{N}\}$$

$$= \sup\{\|F \otimes g)^{(n)}(u)\| : F \in M_p(V^**), g \in M_q(W^*)\|_{\leq 1}, p, q \in \mathbb{N}\} = \|u\|_V.$$ 

Therefore, $\|\theta^{(n)}(u)\|_{cb} = \|u\|_V$ for all $n \in \mathbb{N}$. Consequently, $\theta$ is a completely isometric embedding.

**Remark 6.1.10.** In case that $W$ is complete, we have the completely isometric injection $\theta : V^* \otimes W \hookrightarrow CB(V, W)$.

Just as the Hahn-Banach theorem does in Banach space theory, the following theorem plays a similar role in operator space theory. It is called Arveson-Wittstock-Hahn-Banach theorem. This theorem implies Corollary 2.1.3, one version of classical Hahn-Banach theorem. To see this, take $V$ as a normed subspace of $W$ and let $H = \mathcal{F}(C^* \mathbb{R})$. Then $B(H) = B(\mathcal{F}) = \mathcal{F}$, and we get Corollary 2.1.3.

**Theorem 6.1.11.** (Arveson-Wittstock-Hahn-Banach theorem) If $V$ is a subspace of an operator space $W$ and $H$ is a Hilbert space, then any complete contraction $\varphi : V \hookrightarrow B(H)$ has a completely contractive extension $\Phi : W \hookrightarrow B(H)$.

**Definition 6.1.12.** Let $V$ and $W$ be operator spaces, and $\varphi : V \hookrightarrow W$ a complete quotient mapping. Then a completely bounded mapping $\varphi' : W \hookrightarrow V$ is said to be a lifting for $\varphi$ if $\varphi \circ \varphi' = id_W$.

**Theorem 6.1.13.** Suppose that $V, V_1, W$ and $W_1$ are operator spaces. Given complete contractions $\varphi : V \hookrightarrow V_1$ and $\psi : W \hookrightarrow W_1$, the corresponding mapping $\varphi \otimes \psi : V \otimes W \hookrightarrow V_1 \otimes W_1$ extends to a complete contraction $\varphi \otimes \psi : V \otimes W \hookrightarrow V_1 \otimes W_1$. If $\varphi$ and $\psi$ are complete isometry (resp. complete quotient mappings with completely contractive liftings), then $\varphi \otimes \psi$ is a complete isometry (resp. a complete quotient mapping).

**Proof.** It suffices to show all the cases for $\varphi \otimes \psi : V \otimes W \hookrightarrow V_1 \otimes W_1$. First, we suppose that $\varphi$ and $\psi$ are complete contractions. If $f_1 \in M_p(V_1^*) = CB(V_1, M_p)$ and $g_1 \in M_q(W_1^*) = CB(W_1, M_q)$ are complete contractions, then it is also the case for $f_1 \circ \varphi \in M_p(V^*) = CB(V, M_p)$ and $g_1 \circ \psi \in M_q(W^*) = CB(W, M_q)$. Thus, if
$u \in M_n(V \otimes_v W)$ and we take the suprema over all such complete contractions, then

$$\|(\varphi \otimes \psi)^{(n)}(u)\|_\nu = \sup \{ \|(f_1 \otimes g_1)^{(n)}(u)\| : f_1 \in M_p(V_1^*)_{1 \leq 1}, g_1 \in M_q(W_1^*)_{1 \leq 1}, p, q \in \mathbb{N} \}
= \sup \{ \|(f \otimes g)^{(n)}(u)\| : f \in M_p(V_1^*)_{1 \leq 1}, g \in M_q(W_1^*)_{1 \leq 1}, p, q \in \mathbb{N} \}
\leq \sup \{ \|(f \otimes g)^{(n)}(u)\| : f_1 \in M_p(V_1^*)_{1 \leq 1}, g_1 \in M_q(W_1^*)_{1 \leq 1}, p, q \in \mathbb{N} \} = \|u\|_\nu .$$

Thus, $\varphi \otimes \psi$ is a complete contraction.

Next, we suppose that $\varphi$ and $\psi$ are complete isometries. Then we can identify $V$ and $W$ with operator subspaces of $V_1$ and $W_1$, respectively. Under these identifications, $(\varphi \otimes \psi)^{(n)} : M_n(V \otimes W) \rightarrow M_n(V_1 \otimes W_1)$ is just the inclusion mapping. Let $u \in M_n(V \otimes W)$. Given $\epsilon > 0$, we choose complete contractions $f \in M_p(V^*) \cong CB(V, M_p)$ and $g \in M_q(W^*) \cong CB(W, M_q)$ such that $\|(f \otimes g)^{(n)}(u)\| > \|u\| - \epsilon$. By Arveson-Wittstock-Hahn-Banach theorem, we can extend $f$ and $g$ to complete contractions $f_1 \in M_p(V_1^*)$ and $g_1 \in M_q(W_1^*)$, respectively. Considering that $(\varphi \otimes \psi)^{(n)} : M_n(V \otimes W) \rightarrow M_n(V_1 \otimes W_1)$ is the inclusion mapping, we have

$$\|(\varphi \otimes \psi)^{(n)}(u)\|_\nu \geq \|(f_1 \otimes g_1)^{(n)}(u)\| = \|(f \otimes g)^{(n)}(u)\| > \|u\|_\nu - \epsilon .$$

Since $\epsilon$ is arbitrary, we have $\|(\varphi \otimes \psi)^{(n)}(u)\|_\nu \geq \|u\|_\nu$. The inequality in other direction is obvious from the first part. We thus conclude that $\varphi \otimes \psi$ is completely isometric.

Finally, let $\varphi$ and $\psi$ be complete quotient mappings with complete contractive liftings $\varphi'$ and $\psi'$, respectively. By definition, obviously, $\varphi$ and $\psi$ are completely contractive. So, by the above argument, $\varphi \otimes \psi$ and $\varphi' \otimes \psi'$ are complete contractions. Note that $(\varphi \otimes \psi) \circ (\varphi' \otimes \psi') = id_{V_1 \otimes W_1}$. It follows that $\varphi' \otimes \psi'$ is a completely contractive lifting of $\varphi \otimes \psi$ and $\varphi \otimes \psi$ is a complete quotient mapping. □

**Theorem 6.1.14.** Given operator subspaces $V \subseteq B(H)$ and $W \subseteq B(K)$, the corresponding mapping $V \otimes W \rightarrow B(H \otimes_2 K)$ is a complete isometry.

**Proof.** Let $\varphi : V \rightarrow B(H)$ and $\psi : W \rightarrow B(K)$ be the inclusion mappings. By Theorem 6.1.13, $\varphi \otimes \psi$ is an operator space embedding from $V \otimes W$ into $B(H) \otimes B(K)$. So, it suffices to show that the embedding $B(H) \otimes B(K) \rightarrow B(H \otimes_2 K)$ is completely isometric.
Firstly, by Theorem 6.1.9, the embedding $B(H) \otimes B(K) \hookrightarrow CB(B(H)_*, B(K))$ is completely isometric. Secondly, $CB(B(H)_*, B(K)) \cong (B(H)_* \otimes B(K)_*)^*$ completely isometrically. So, $B(H) \otimes B(K) \hookrightarrow (B(H)_* \otimes B(K)_*)^*$ is completely isometric. By Fubini Theorem (Theorem 5.6.2), we have operator space identification $(B(H)_* \otimes B(K)_*)^* \cong B(H \otimes_2 K)$. Thus we are done. □

**Corollary 6.1.15.** Let $V, W$ and $X$ be operator spaces. Then we have the following complete isometries:

$$V \otimes W \cong W \otimes V \quad \text{and} \quad (V \otimes W) \otimes X \cong V \otimes (W \otimes X).$$

**Proof.** Let $V$ and $W$ be operator subspaces of $B(H)$ and $B(K)$, respectively. Note that $H \otimes_2 K \cong K \otimes_2 H$ as Hilbert spaces. Thus $B(H \otimes_2 K) \cong B(K \otimes_2 H)$ as operator spaces. It follows by Theorem 6.1.14 that $V \otimes W \cong W \otimes V$.

Similarly, we can prove the second identification. □

**Corollary 6.1.16.** For any operator space $V$ and $n \in \mathbb{N}$, we have a natural complete isometry $M_n(V) \cong M_n \otimes_v V$.

**Proof.** First, we have a linear space isomorphism $M_n(V) \cong M_n \otimes_v V$ defined by

$$\varphi : M_n(V) \longrightarrow M_n \otimes_v V, \quad v = [v_{i,j}] \mapsto \sum_{i,j=1}^{n} E_{i,j} \otimes v_{i,j},$$

where $E_{i,j}$ is the element in $M_n$ with $(i, j)$-th element to be 1 and all other elements to be zero. So, it suffices to show that, at each level $m$, the map $\varphi^{(m)}$ is an isometry. This can be seen from the following diagram

$$\begin{array}{ccc}
CB^\sigma(V^*, M_n) & \overset{i}{\longrightarrow} & CB(V^*, M_n) \\
\psi \uparrow & & \theta \uparrow \\
M_n(V) & \overset{\varphi}{\longrightarrow} & M_n \otimes_v V \\
\end{array}$$

where $i : CB^\sigma(V^*, M_n) \longrightarrow CB(V^*, M_n)$ is an operator space embedding, $\psi : M_n(V) \longrightarrow CB^\sigma(V^*, M_n)$, defined by $\psi(v)(f) = (f(v_{i,j}))_n$ for all $f \in V^*$ and $v \in M_n(V)$, is a complete isometry, and $\theta : M_n \otimes_v V \longrightarrow CB(V^*, M_n)$, determined by $\theta(T \otimes v)(f) = f(v)T$ for all $f \in V^*, v \in V$ and $T \in M_n$, is an operator space embedding. It is easy to see that $\varphi \circ \theta = \psi$. So, the diagram is commutative. □

**Remark 6.1.17.** In case that $V$ is complete, we have $M_n(V) \cong M_n \otimes_v V$. This is because $M_n(V)$ is complete if and only if $V$ is complete by Proposition 4.1.6. So, it
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follows from Corollary 6.1.16 that $M_n(V) \cong M_n \otimes V$ is a complete isometry if $V$ is complete.

**Theorem 6.1.18.** Both $\| \cdot \|_V$ and $\| \cdot \|_\wedge$ are cross norms.

**Proof.** Let $V$ and $W$ be operator spaces. Then there are Hilbert spaces $H$ and $K$ such that $V \rightarrow B(H)$ and $W \rightarrow B(K)$ are completely isometric inclusions. By Theorem 6.1.14, the corresponding mapping $V \otimes_V W \rightarrow B(H \otimes_2 K)$ is also a completely isometric inclusion. So, for all $p, q \in \mathbb{N}$, $M_{pq}(V \otimes_V W) \rightarrow M_{pq}(B(H \otimes_2 K))$ is an isometric injection. But the latter can be identified with $B(H^p \otimes_2 K^q)$. It follows that

$$M_{pq}(V \otimes_V W) \rightarrow B(H^p \otimes_2 K^q)$$

is an isometric injection.

On the other hand, for all Hilbert spaces $H_i$ and $K_i$, and bounded operators $b_i : H_i \rightarrow K_i$ ($i = 1, 2$), the tensor product $b_1 \otimes b_2 : B(H_1 \otimes_2 H_2) \rightarrow B(K_1 \otimes_2 K_2)$ satisfies $\|b_1 \otimes b_2\| = \|b_1\| \|b_2\|$. So, it follows from Equality (41) that $\|v \otimes w\|_V = \|v\| \|w\|$ for all $v \in M_p(V)$ and $w \in M_q(W)$. This shows that $\| \cdot \|_V$ is a cross norm.

To see that $\| \cdot \|_\wedge$ is also a cross norm, note that it is the largest subcross norm. Thus, from the following relations

$$\|v\| \|w\| = \|v \otimes w\|_V \leq \|v \otimes w\|_\wedge \leq \|v\| \|w\|,$$

we get $\|v \otimes w\|_\wedge = \|v\| \|w\|$ for all $v \in M_p(V)$ and $w \in M_q(W)$ ($p, q \in \mathbb{N}$). \qed

6.2. Banach Dual of Injective Tensor Products

In this section, we will discuss the Banach dual of injective tensor products. All definitions and properties present in this section are cited from [19] and [12].

Let $X$ and $Y$ be Banach spaces. Let $B_X^*$ and $B_Y^*$ denote the topological spaces of the unit balls of Banach duals $X^*$ and $Y^*$ carrying theire weak* topologies, respectively. By Alaoglu theorem, both $B_X^*$ and $B_Y^*$ are compact. So, the product space $B_X^* \times B_Y^*$ under the product topology is also compact. We still use $B_X^* \times B_Y^*$ to denote the corresponding topological space. Let $C(B_X^* \times B_Y^*)$ denote the normed space of all continuous functions $F$ on $B_X^* \times B_Y^*$ with $\|F\| = \sup\{ |F(f)| : f \in B_X^* \times B_Y^* \}$.

Then $C_0(B_X^* \times B_Y^*) = C(B_X^* \times B_Y^*)$, where $C_0(B_X^* \times B_Y^*)$ denotes the subspace of $C(B_X^* \times B_Y^*)$ whose elements being functions vanishing at infinity.
For any \( u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes \lambda Y, n \in \mathbb{N} \), we can regard it as a bilinear form on \( X^* \times Y^* \) defined by
\[
\langle u, (\varphi, \psi) \rangle = \sum_{i=1}^{n} \varphi(x_i) \psi(y_i).
\] (42)
If we restrict \( u \) on the topological space \( B_{X^*} \times B_{Y^*} \), then \( u \) is continuous. Moreover, \( u \) is bounded satisfying
\[
\|u\| = \sup \{|u(f)| : f \in B_{X^*} \times B_{Y^*}\} = \sup \{|u(\varphi, \psi)| : \varphi \in B_{X^*}, \psi \in B_{Y^*}\} = \|u\|_{\lambda}.
\]
Thus, every \( u \in X \otimes \lambda Y \) can be viewed as an element of \( C(B_{X^*} \times B_{Y^*}) \). So \( X \otimes \lambda Y \) can be embedded in \( C(B_{X^*} \times B_{Y^*}) \) isometrically. Since the later one is complete, the embedding can be extended isometrically to the completion
\[
X \otimes \lambda Y \subseteq C(B_{X^*} \times B_{Y^*}).
\]
Therefore, any element \( u \in X \otimes \lambda Y \) can be viewed as a continuous function on \( B_{X^*} \times B_{Y^*} \).

**Lemma 6.2.1.** For any bounded linear form \( \tilde{B} \) on \( X \otimes \lambda Y \), there is a finite regular Borel measure \( \mu \) on \( B_{X^*} \times B_{Y^*} \) such that
\[
\langle \tilde{B}, u \rangle = \int_{B_{X^*} \times B_{Y^*}} u(\varphi, \psi) d\mu(\varphi, \psi).
\] (43)
for all \( u \in X \otimes \lambda Y \). Moreover, \( \|\tilde{B}\| = \|\mu\| \), where \( u \) is defined in (42), and \( \|\mu\| = |\mu|(X) \) is the variation norm of \( \mu \).

**Proof.** Since \( X \otimes \lambda Y \) is closed in \( C(B_{X^*} \times B_{Y^*}) \), by Hahn-Banach theorem, every bounded functional \( \tilde{B} \) on \( X \otimes \lambda Y \) can be extended to \( C(B_{X^*} \times B_{Y^*}) \), and the extension preserves the norm. We still use \( \tilde{B} \) to denote the extension. On the other hand, since \( C_0(B_{X^*} \times B_{Y^*}) = C(B_{X^*} \times B_{Y^*}) \), by Riesz representation theorem, we have \( C(B_{X^*} \times B_{Y^*})^* \cong M(B_{X^*} \times B_{Y^*}) \), where \( M(B_{X^*} \times B_{Y^*}) \) is the normed space of all finite complex regular Borel measures on \( B_{X^*} \times B_{Y^*} \). Moreover, \( \|\tilde{B}\| = \|\mu\| \), where \( \mu \) corresponds \( \tilde{B} \) by the map \( \tau \) defined in Theorem 2.5.1. So, we are done.

**Lemma 6.2.2.** Let \( X \) and \( Y \) be Banach spaces. Let \( B \) be a bilinear form on \( X \times Y \), and \( \tilde{B} \) be its corresponding linear functional on \( X \otimes Y \). Then \( \tilde{B} \) is bounded on \( X \otimes \lambda Y \) if and only if there exists a finite regular Borel measure \( \mu \) on the compact...
space $B_{X^*} \times B_{Y^*}$ such that

$$B(x, y) = \int_{B_{X^*} \times B_{Y^*}} \varphi(x) \psi(y) d\mu(\varphi, \psi)$$

(44)

for all $x \in X$ and $y \in Y$. Furthermore, the norm of $\bar{B}$ is given by

$$\|\bar{B}\| = \inf \|\mu\|,$$

(45)

where the infimum is taken over all measures $\mu$ determining $B$ as in (44), and the infimum is attained.

**Proof.** First, we suppose that $\bar{B}$ is a bounded linear functional on $X \otimes^\alpha Y$. By Lemma 6.2.1, there is a finite regular Borel measure $\mu$ on $B_{X^*} \times B_{Y^*}$ such that formula (43) holds, and $\|\bar{B}\| = \|\mu\|$. Let $u = x \otimes y$, we have

$$B(x, y) = \langle \bar{B}, x \otimes y \rangle = \int_{B_{X^*} \times B_{Y^*}} \varphi(x) \psi(y) d\mu(\varphi, \psi).$$

Thus, formula (44) holds and $\|\bar{B}\| = \inf \|\mu\|$, where the infimum is taken over all measures $\mu$ determining $B$ as in (44), and the infimum is attained.

On the other hand, for any finite regular Borel measure $\mu$ on $B_{X^*} \times B_{Y^*}$, we can define a linear functional $\bar{B}$ on $X \otimes^\alpha Y$ by (43). The corresponding bilinear form $B$ satisfies (44). We have by (43)

$$\|\bar{B}\| = \sup_{\|u\| \leq 1} |\langle \bar{B}, u \rangle| = \sup_{\|u\| \leq 1} \left| \int_{B_{X^*} \times B_{Y^*}} u(\varphi, \psi) d\mu(\varphi, \psi) \right|$$

$$\leq \sup_{\|u\| \leq 1} \int_{B_{X^*} \times B_{Y^*}} |u(\varphi, \psi)| d\|\mu(\varphi, \psi)\| \leq \int_{B_{X^*} \times B_{Y^*}} d\|\mu(\varphi, \psi)\| = \|\mu\|.$$ 

Thus $\bar{B}$ is bounded. Taking infimum on all those measure $\mu$, we have $\|\bar{B}\| \leq \inf \|\mu\|$. Together with the first part, we see that $\|\bar{B}\| = \inf \|\mu\|$ and the infimum is attained. 

Let $X$ and $Y$ be Banach spaces. A bilinear form $B$ on $X \times Y$ is said to be **integral** if its linearization $\bar{B}$ is a bounded linear functional on the injective tensor product $X \otimes^\alpha Y$. The integral norm of $B$ is defined by $\|B\|_I = \inf \|\mu\|$, where the infimum is taken over all the finite regular Borel measures $\mu$ on $B_{X^*} \times B_{Y^*}$ satisfying (44). Then all the integral bilinear forms on $X \times Y$ form a normed space with the integral norm.
Its completion is denoted by \( B_I(X \times Y) \), called the space of integral bilinear forms. The following is a direct consequence of Lemma 6.2.2.

**Theorem 6.2.3.** We have the following isometric identification

\[
(X \otimes^I Y)^* \cong B_I(X \times Y).
\]  

**Definition 6.2.4.** Let \( X \) and \( Y \) be Banach spaces. For every operator \( T : X \to Y \), define \( B_T : X \times Y^* \to \mathbb{C} \) by \( B_T(x, \psi) = (Tx, \psi) \) for all \( x \in X \) and \( \psi \in Y^* \). Then \( B_T \) is a bilinear form. A linear operator \( T : X \to Y \) is said to be integral if the corresponding bilinear form \( B_T \) is integral, and we define the integral norm of \( T \), denoted by \( \|T\|_I \), to be the integral norm of the bilinear form \( B_T \). The space of integral operators from \( X \) into \( Y \) with this norm is denoted by \( I(X, Y) \).

**Proposition 6.2.5.** Let \( X \) and \( Y \) be Banach spaces. Then for every \( T \in I(X, Y) \), we have \( \|T\| \leq \|T\|_I \).

**Proof.** For all \( x \in X \) and \( f \in Y^* \), we have

\[
|\langle Tx, f \rangle| = |B_T(x, f)| = \left| \int_{B_{X^*} \times B_{Y^*}} \varphi(f) \psi(x) d\mu(\varphi, \psi) \right| 
\leq \int_{B_{X^*} \times B_{Y^*}} |\varphi(f)| |\psi(x)| d\mu(\varphi, \psi) \leq \|x\| \|f\| \|\mu\|,
\]

where \( \mu \), corresponding to \( B_T \) as in (44), is a finite regular Borel measure on \( B_{X^*} \times B_{Y^*} \). So \( \|T\| \leq \|\mu\| \). Taking infimum, we have \( \|T\| = \|B_T\| \leq \|T\|_I \). \( \square \)

By Proposition 6.2.5, the correspondence \( T \mapsto B_T \) determines an isometric embedding \( I(X, Y) \hookrightarrow B_I(X \times Y^*) \). Moreover, we have the following theorem (see [19, Proposition 3.22]).

**Theorem 6.2.6.** Let \( X \) and \( Y \) be two Banach spaces. A bilinear form \( B \) on \( X \times Y \) is integral if and only if the associated operator \( T_B : X \to Y^* \), defined by \( \langle T_Bx, y \rangle = B(x, y) \) for all \( x \in X \) and \( y \in Y \), is integral. The mapping \( B \mapsto T_B \) is an isometric isomorphism from \( B_I(X \times Y) \) onto \( I(X, Y^*) \).

This theorem, together with Theorem 6.2.3, gives a way to characterize the Banach dual of the injective tensor product of Banach spaces \( X \) and \( Y \) in terms of the integral mappings:

\[
(X \otimes^I Y)^* \cong I(X, Y^*).
\]
Definition 6.2.7 and Definition 6.2.9 below give another way to characterize the integral maps. These definitions and corresponding properties are cited from [12] and [14].

**Definition 6.2.7.** Let $X$ and $Y$ be Banach spaces. A linear map $\phi : X \to Y$ is said to be nuclear if it lies in the range of the natural contraction

$$\Phi : X^* \hat{\otimes} Y \to X^* \hat{\otimes} Y \subseteq B(X,Y).$$

(47)

The linear space isomorphism

$$\Phi(X^* \hat{\otimes} Y) \cong (X^* \hat{\otimes} Y)/\ker \Phi$$

induces a quotient norm $\nu_B$ on $\Phi(X^* \hat{\otimes} Y)$. We use $N(X,Y)$ to denote the corresponding Banach space with this norm.

As an example, let us see the nuclear maps from $c_0$ to a Banach space $X$. As proved in [19, Example 2.6], $c_0 \hat{\otimes} X = l_1 \hat{\otimes} X$ can be identified with the space $l_1(X)$ as normed spaces. Under this identification, the canonical mapping $J : c_0 \hat{\otimes} X = l_1(X) \to N(c_0, X)$ associates with the element $x = (x_n)$ of $l_1(X)$ the nuclear map given by $T_x(a) = \sum_{n=1}^{\infty} a_n x_n$ for all $a \in c_0$. It follows that $J$ is injective. Hence we have $N(c_0, X) = l_1 \hat{\otimes} X = l_1(X)$.

**Theorem 6.2.8.** A linear map $\phi : X \to Y$ is nuclear if and only if there is a commutative diagram of bounded maps

$$\begin{array}{ccc}
\ell^\infty & \xrightarrow{\theta_\sigma} & l^1 \\
\downarrow \sigma & & \downarrow \tau \\
X & \xrightarrow{\phi} & Y
\end{array}$$

with $\|\sigma\| \cdot \|\theta_\sigma\| \cdot \|\tau\| < \kappa$ for some positive number $\kappa$, where $\theta_\sigma$ is the diagonal map $(\alpha_i) \mapsto (d_i \alpha_i)$ for some sequence $d = (d_i) \in l^1$.

**Definition 6.2.9.** A linear map $\phi : X \to Y$ is said to be integral if and only if it can be approximated in the point norm topology by nuclear maps $\psi : X \to Y$ with $\nu_B(\psi) < \kappa$ for some $\kappa > 0$. The integral norm of $\phi$, denoted by $\iota_B(\phi)$, is defined to be the infimum of all such $\kappa$. 

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The space of all integral maps from $X$ to $Y$ with integral norm $\iota_B$ defined this way is still denoted by $I(X,Y)$. As shown in [12], in terms of integral maps as defined in Definition 6.2.9, we still have

**Theorem 6.2.10.** Let $X$ and $Y$ be Banach spaces. Then

$$I(X,Y^*) \cong (X \otimes^\lambda Y)^*$$

is an isometry.

Comparing this result with Theorem 6.2.3 and Theorem 6.2.6, we see that the two definitions of integral maps defined above (Definition 6.2.4 and Definition 6.2.9) are equivalent, and $\|\varphi\|_I = \iota_B(\varphi)$ for all integral map $\varphi$. In next section, we will use an approach similar to Definition 6.2.9 to define completely integral maps.

### 6.3. Operator Dual of Injective Tensor Product

Motivated by the Banach space case, the space $I(V,W)$ of completely integral maps for operator spaces $V$ and $W$ is defined in a similar way in [12]. Unlike the Banach space case, the operator space identification $I(V,W^*) \cong (V \otimes^\nu W)^*$ for operator spaces $V$ and $W$ only holds under some conditions. In the following, we first introduce the definition of completely nuclear maps, with which we define completely integral maps for operator spaces.

**Definition 6.3.1.** Let $V$ and $W$ be operator spaces. A linear map $\phi : V \rightarrow W$ is said to be completely nuclear if it lies in the range of the natural complete contraction

$$\Phi : V^* \widehat{\otimes} W \rightarrow V^* \otimes^\nu W \subseteq CB(V,W). \quad (49)$$

The linear space isomorphism

$$\Phi(V^* \widehat{\otimes} W) \cong V^* \otimes^\nu W / \ker \Phi$$

induces a quotient operator space structure on $\Phi(V^* \widehat{\otimes} W)$. We use $N(V,W)$ to denote the corresponding operator space and use $\nu_p(\phi)$ to denote the matrix norm of $\phi \in M_p(N(V,W))$. In case that $p = 1$, we simply write $\nu(\phi)$ for $\nu_1(\phi)$. It is easy to see that $\nu_p(\phi) \geq \|\phi\|_{cb}$ for all $\phi \in M_p(N(V,W))$. 

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Since, as sets, we have the relation:

\[ M_p(N(V, W)) = M_p(\Phi(V^* \hat{\otimes} W)) \subseteq M_p(CB(V, W)). \]  \hspace{1cm} (50)

By the identification \( M_p(CB(V, W)) = CB(V, M_p(W)) \), we may regard \( \phi \in M_p(N(V, W)) \) as a map \( \phi : V \rightarrow M_p(W) \). However, in general, \( M_p(N(V, W)) \neq N(V, M_p(W)) \).

The correct relations are (see [13, page 210])

\[ T_n(N(V, W)) \cong N(V, T_n(W)) \cong N(M_n(V), W). \]

As in [11], we let \( M_\infty, K_\infty \) and \( T_\infty \) denote the operator spaces of the bounded, compact and trace class operators on \( \ell^2 \), respectively. We use the bilinear form

\[ M_\infty \times T_\infty \rightarrow C : (S, c) \mapsto (S, c) = \sum S_{i,j}c_{i,j} \]

to identify \( T_\infty \) with the predual of \( M_\infty \), where \( S = (S_{i,j}) \in M_\infty, c = (c_{i,j}) \in T_\infty \) (i, j \in \mathbb{N}) .

We also let \( H_\infty \) denote the Hilbert space of Hilbert-Schmidt operators on \( \ell^2 \), the space of the matrices \( a = [a_{i,j}] \) with Hilbert-Schmidt norm \( \|a\| = (\sum |a_{i,j}|^2)^{1/2} < \infty \).

We may use the bilinear pairing

\[ H_\infty \times H_\infty \rightarrow C : (a, b) \mapsto \langle a, b \rangle = \sum a_{i,j}b_{i,j} \]

to identify \( H_\infty^* \) with \( H_\infty \) as Hilbert spaces. Note that any square matrix in \( H_\infty \) (and \( H_\infty^* \) as well) is a \( \infty \times \infty \) matrix. From [13, Section 10.1], we have operator space identifications \( M_\infty \cong M_{\infty \times \infty, 1} \cong M_{1, \infty \times \infty} \). By regarding a square matrix in \( H_\infty \) as an \( (\infty \times \infty) \times 1 \) (column) matrix, and a square matrix in \( H_\infty^* \) as a \( 1 \times (\infty \times \infty) \) (row) matrix, we may give \( H_\infty \) and \( H_\infty^* \) the (distinct) operator space structures

\[ H_\infty \cong M_{\infty \times \infty, 1} \cong M_{\infty \times \infty}(M_1), \quad H_\infty^* \cong M_{1, \infty \times \infty} \cong M_1(M_{\infty \times \infty}). \]

In particular, we have

\[ M_{1,p}(H_\infty) \cong M_{\infty \times \infty, p} \cong M_{\infty \times \infty}(M_p), \quad M_{p,1}(H_\infty^*) \cong M_{p, \infty \times \infty} \cong M_p(M_{\infty \times \infty}) \]

for all \( p \in \mathbb{N} \). With these notations and assumptions, we have (see [12])

**Theorem 6.3.2.** Suppose that \( V \) and \( W \) are operator spaces. Then a linear map \( \phi : V \rightarrow M_p(W) \) lies in \( M_p(N(V, W)) \) if and only if there is a commutative diagram
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of maps

\[ M_\infty \xrightarrow{\theta_{a,b}} M_p(T_\infty) \]

\[ \sigma \uparrow \quad \tau \downarrow \]

\[ V \xrightarrow{\phi} M_p(W) \]

where \( a \in M_{p,1}(H_\infty^*) \) and \( b \in M_{1,p}(H_\infty) \), \( \theta_{a,b} : M_\infty \to M_p(Y_\infty) \) maps \( T \) to \( aTb \) for all \( T \in M_\infty \), and \( \sigma \) and \( \tau \) are completely bounded linear maps. Furthermore, we have

\[ \nu_p(\phi) = \inf \{ \| \tau \|_{ob} \| a \| \| b \| \| \sigma \|_{ob} \} , \]

where the infimum is taken over all such diagrams.

Here we want to point out that “completely nuclear” is a very strong restriction on a map \( \phi : V \to W \) for operator spaces \( V \) and \( W \). As an example, we have the following proposition (see [12, Proposition 2.3]).

**PROPOSITION 6.3.3.** Let \( V \) be an operator space. Then the identity map \( id_V : V \to V \) is completely nuclear if and only if \( V \) is finite dimensional.

**DEFINITION 6.3.4.** Let \( V \) and \( W \) be operator spaces. A linear map \( \phi : V \to W \) is said to be completely integral if it is a point-norm limit of completely nuclear maps \( \psi : V \to W \) with \( \nu(\psi) = \nu_1(\psi) < \kappa \) for some constant \( \kappa > 0 \). We define the integral norm \( \iota(\phi) \) to be the infimum of such constants \( \kappa \).

By definition, if there is some net \( \psi_\alpha \in N(V,W) \) converging to \( \phi \) in the point-norm topology with \( \nu(\psi_\alpha) < \kappa \), then we have \( \iota(\phi) \leq \kappa \). On the other hand, if \( \iota(\phi) \leq \kappa \), then by definition, there is a net \( \psi_\alpha \in N(V,W) \) converging to \( \phi \) in the point-norm topology with \( \nu(\psi_\alpha) \leq \kappa \). The following proposition is a direct consequence of Theorem 6.3.2.

**PROPOSITION 6.3.5.** Given operator spaces \( V \) and \( W \). A map \( \phi : V \to W \) is completely integral if and only if there is a net \( \psi_\alpha : V \to W \) converging to \( \phi \) in the point-norm topology such that, for each \( \psi_\alpha \), the diagram

\[ M_\infty \xrightarrow{\theta_{a,b}} M_p(T_\infty) \]

\[ \sigma_\alpha \uparrow \quad \tau_\alpha \downarrow \]

\[ V \xrightarrow{\psi_\alpha} M_p(W) \]
commutes with some \( \sigma_a, \tau_a \) and \( \theta_{a,b} \) defined as in Theorem 6.3.2 satisfying

\[
\|\sigma_a\| \|a\| \|b\| \|\tau_a\| \|b\| < \kappa
\]

for some \( \kappa_a > 0 \).

In the case that a map \( \phi : V \to W \) is completely integral, we also say that the diagram

\[
\begin{array}{ccc}
M_{\infty} & \xrightarrow{\theta_{a,b} \cdot \phi} & M_p(T_{\infty}) \\
\downarrow{\sigma_a} & & \downarrow{\tau_a} \\
V & \xrightarrow{\phi} & M_p(W)
\end{array}
\]

is approximately commutative in the point-norm topology.

We denote by \( I(V, W) \) the linear space of all integral maps with the integral norm. To define a matrix norm on \( I(V, W) \), we view the elements of \( M_p(I(V, W)) \) as linear maps from \( V \) into \( M_p(W) \) defined by \( \phi(v) = [\phi_{i,j}(v)] \) for each \( \phi = [\phi_{i,j}] \in M_p(I(V, W)) \) and all \( v \in V \). It is easy to see ([12, page 178]) that a linear map \( \phi : V \to M_p(W) \) lies in \( M_p(I(V, W)) \) if and only if it is the point-norm limit of completely nuclear maps \( \psi : V \to M_p(W) \) with \( \nu(\psi) < \kappa \) for some constant \( \kappa > 0 \). We define the integral matrix norm \( \iota_p(\phi) \) of \( \phi \) to be the infimum of such constants \( \kappa \) that there is a net of completely nuclear maps \( \psi : V \to M_p(W) \) with \( \nu_p(\psi) < \kappa \) converging to \( \phi \) in the point-norm topology. Therefore, we have a norm space identification \( M_p(I(V, W)) \cong I(V, M_p(W)) \).

**Proposition 6.3.6.** Let \( V \) and \( W \) be operator spaces and \( p \in \mathbb{N} \). Then we have the following set inclusion relations:

\[
M_p(N(V,W)) \subseteq M_p(I(V,W)) \subseteq M_p(CB(V,W)).
\]

Moreover, \( \iota_p(\psi) \leq \nu_p(\psi) \) for all \( \psi \in M_p(N(V,W)) \) and \( \|\psi\|_{cb} \leq \iota_p(\psi) \) for all \( \psi \in M_p(I(V,W)) \).

**Proof.** Suppose that \( \psi \in M_p(N(V,W)) \). Let \( \nu_p(\psi) = \kappa \). For any \( \epsilon > 0 \), we choose the net \( \psi_\alpha \equiv \psi \). Then \( \psi_\alpha \to \psi \) in point-norm topology with \( \nu_p(\psi_\alpha) < \kappa + \epsilon \) for any \( \epsilon > 0 \). Thus, \( \iota_p(\psi) \leq \kappa + \epsilon \). Since \( \epsilon > 0 \) is arbitrary, we have \( \iota_p(\psi) \leq \kappa \). Thus \( \psi \in M_p(I(V,W)) \), i.e., \( M_p(N(V,W)) \subseteq M_p(I(V,W)) \) and \( \iota_p(\psi) \leq \nu_p(\psi) \) for all \( \psi \in M_p(N(V,W)) \).
Now we suppose that $\psi \in M_p(I(V,W))$ and $\iota_p(\psi) = \kappa$. Then for any $\epsilon > 0$, there is a net $\psi_\alpha \in M_p(N(V,W))$ converging to $\psi$ in the point-norm topology with $\nu_p(\psi_\alpha) < \kappa + \epsilon$. Note that, $\|\psi_\alpha\|_{cb} \leq \nu_p(\psi_\alpha) < \kappa + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\|\psi_\alpha\|_{cb} \leq \kappa = \iota_p(\psi)$. From here, we claim that $\|\psi\|_{cb} \leq \iota_p(\psi)$ and so $\psi \in M_p(CB(V,W)).$ Otherwise, there would exist $n \in \mathbb{N}$ and $\nu_0 \in M_n(V)$ with $\|\nu_0\| \leq 1$ such that $\|\psi^{(n)}(\nu_0)\| > \iota_p(\psi)$. Thus, there exists an $\alpha_0$ with $\|\psi_{\alpha_0}^{(n)}(\nu_0)\| > \iota_p(\psi)$. But then,

$$\|\psi_{\alpha_0}\|_{cb} \geq \|\psi_{\alpha_0}^{(n)}(\nu_0)\| > \iota_p(\psi).$$

This is a contradiction! It follows that $M_p(I(V,W)) \subseteq M_p(CB(V,W))$ and $\iota_p(\psi) \geq \|\psi\|_{cb}$ for all $\psi \in M_p(I(V,W)).$ \hfill $\square$

**Theorem 6.3.7.** Let $V$ and $W$ be operator spaces. Then the space $I(V,W)$ with the above matrix norm is an operator space.

**Proof.** First, for any $\phi \in M_p(I(V,W))$, $\psi \in M_q(I(V,W))$ and $\epsilon > 0$, there are completely nuclear maps $\phi_\gamma \in M_p(N(V,W))$ with $\nu_p(\phi_\gamma) < \iota_p(\phi) + \epsilon$ and $\psi_\delta \in M_q(N(V,W))$ with $\nu_q(\psi_\delta) < \iota_q(\psi) + \epsilon$ such that $\phi_\gamma$ and $\psi_\delta$ converge to $\phi$ and $\psi$ in point-norm, respectively. Thus the net of completely nuclear maps $\{\phi_\gamma \oplus \psi_\delta\}$ converges to $\phi \oplus \psi$ in the point-norm topology with

$$\nu_{p+q}(\phi_\gamma \oplus \psi_\delta) = \max\{\nu_p(\phi_\gamma), \nu_q(\psi_\delta)\} < \max\{\nu_p(\phi), \nu_q(\psi)\} + \epsilon.$$

It follows from Definition 6.3.4 that $\phi \oplus \psi \in M_{p+q}(I(V,W))$ with

$$\iota_{p+q}(\phi \oplus \psi) \leq \max\{\iota_p(\phi), \iota_q(\psi)\}.$$

Since the inverse is obvious, we get $M1$.

To prove $M2$, let $\alpha \in M_{n,p}, \beta \in M_{p,n}$. Then the net of completely nuclear maps $\{\alpha \phi_\gamma \beta\}$ converges to $\alpha \phi \beta$ in the point-norm topology with

$$\nu_n(\alpha \phi_\gamma \beta) \leq \|\alpha\| \nu_p(\phi_\gamma) \|\beta\| < \|\alpha\| \|\nu_p(\phi) + \epsilon\| \|\beta\|.$$

It follows from Definition 6.3.4 that $\alpha \phi \beta \in M_p(I(V,W))$ and

$$\iota_n(\alpha \phi \beta) \leq \|\alpha\| \iota_p(\phi) \|\beta\|.$$

Thus, $M2$ holds. \hfill $\square$
**Theorem 6.3.8.** Suppose that \( \phi : V \to M_p(W) \) is a linear map such that the diagram

\[
\begin{array}{ccc}
M_\infty & \xrightarrow{\theta_{a,b}} & M_p(T_\infty) \\
\sigma & \downarrow \tau_a & \\
V & \xrightarrow{\phi} & M_p(W^{**})
\end{array}
\]

(55)

satisfying

\[
\|\sigma\|_{cb}\|a\|\|b\|\|\tau\|_{cb} < \kappa
\]

approximately commutes in the point-weak topology in the sense that \( |\phi(v)(f) - \tau_a \circ \theta_{a,b} \circ \sigma(a)(v)(f)| \to 0 \) for all \( v \in V \) and \( f \in (M_p(W^{**}))^* \). Then we must have \( \phi \in M_p(I(V,W)) \) with \( \psi_p(\phi) \leq \kappa \).

Given an inclusion of operator spaces \( W \subseteq W_1 \), the corresponding map \( I(V,W) \to I(V,W_1) \) need not be completely isometric. However, we have the following proposition (see [12, Proposition 3.3]).

**Theorem 6.3.9.** For any operator spaces \( V \) and \( W \), the natural embedding \( I(V,W) \hookrightarrow I(V,W^{**}) \) is completely isometric.

**Proof.** First, by Theorem 5.5.8 and Remark 6.1.5, we have

\[
CB(M_p, W^{**}) \cong (M_p \otimes W)^{**} \cong M_p(W)^{**} \cong M_p(W^{**}) \cong CB(M_p^{**}, W). \tag{57}
\]

Given \( \phi : V \to M_p(W) \) with \( \psi_p(\phi) < \kappa \), if we regard \( \phi \) as a map from \( V \) into \( M_p(W^{**}) \), then the diagram (55) satisfying (56) which approximately commutes in the point-norm topology. Given

\[
\tau \in CB(M_p^{**}, W^{**}) = M_n(W^{**}),
\]

we may use Equality (57) and the Bipolar Theorem to approximate \( \tau \) in the weak* topology by maps \( \tau' : M_p^{**} \to W \) with \( \|\tau\|_{cb} = \|\tau'\|_{cb} \). The compositions

\[
\tau'_p \circ \theta_{a,b} \circ \sigma : V \to M_p(W)
\]

approximate \( \phi \) in the point-weak topology. By Theorem 6.3.8, we conclude that \( \psi_p(\phi) \leq \kappa \) when \( \phi \) is regarded as a map from \( V \) into \( M_p(W) \). \( \square \)
We have a natural commutative diagram of complete contractions

\[
\begin{array}{ccc}
V^* \otimes W^* & \xrightarrow{\theta} & (V \otimes W)^* \\
\downarrow S & & \downarrow S \\
N(V, W^*) & \xrightarrow{l_1} & I(V, W^*) & \xrightarrow{l_2} & CB(V, W^*)
\end{array}
\]

where \(l_1\) and \(l_2\) are the inclusion maps, \(S\) is the identification map, which is a complete isometry. \(\Phi : V \otimes W \rightarrow V \otimes W\) and \(\Phi : V^* \otimes W^* \rightarrow N(V, W^*) = V^* \otimes W^*/\ker \Phi\) are the canonical maps. They are both complete contractions. Since \(\Phi\) has dense range, \(\Phi^*\) is an injective contraction. The map \(\theta\) is determined by

\[
\theta(f \otimes g)(v \otimes w) = f(v)g(w)
\]

for \(f \in V^*, g \in W^*, v \in V\) and \(w \in W\). To see that \(\theta\) is a complete contraction, note that \(V \otimes W \rightarrow (V^* \otimes W^*)^*\) is a completely isometric injection (See [22, Theorem 3.2.8]). So, the dual map \((V^* \otimes W^*)^{**} \rightarrow (V \otimes W)^*\) is a complete contraction.

With the restriction on \(V^* \otimes W^*\), we see that \(V^* \otimes W^* \rightarrow (V \otimes W)^*\) is a complete contraction, which is exactly the map \(\theta\).

The diagram is commutative since it is immediate that \(S(\Phi(F)) = \Phi^*(\theta(F))\) for \(F = f \otimes g\) with all \(f \in V^*\) and \(g \in W^*\). By linearity and continuity, we find that this relation holds for all \(F \in V^* \otimes W^*\).

**Theorem 6.3.10.** Given operator spaces \(V\) and \(W\), there is a unique complete contraction \(S_I : I(V, W^*) \rightarrow (V \otimes W)^*\) for which the following diagram commutes:

\[
\begin{array}{ccc}
V^* \otimes W^* & \xrightarrow{\theta} & (V \otimes W)^* \\
\downarrow S_I & & \downarrow S \\
N(V, W^*) & \xrightarrow{l_1} & I(V, W^*) & \xrightarrow{l_2} & CB(V, W^*)
\end{array}
\]

**Proof.** Note that for any \(F \in M_p((V \check{\otimes} W)^*)\), we have

\[
\|F(u)\| \leq \|F\|_{\alpha \beta}\|u\|_{V} \leq \|F\|_{\alpha \beta}\|u\|_{\Lambda}
\]

for all \(u \in V \check{\otimes} W\). Moreover, \(V \check{\otimes} W\) is dense in \(V \otimes W\). Thus, any element \(F\) in \(M_p((V \otimes W)^*)\) can be identified with an element in \(M_p((V^* \otimes W)^*)\). Let \(p \in \mathbb{N}\) be
fixed. Our task is to show that

$$\| S^{(p)}(\phi) \|_{CB(V \hat{\otimes} W, M_p)} \leq \epsilon_p(\phi)$$

for all $\phi \in M_p(I(V, W^*))$. Then it follows that $S$ maps $I(V, W^*)$ into $(V \hat{\otimes} W)^*$ and the restriction $S_I$ of $S$ to $I(V, W^*)$ is a complete contraction.

For any $\phi \in M_p(I(V, W^*))$ with $\epsilon_p(\phi) \leq M$ for some constant $M > 0$, by definition, there is a net $\psi_\alpha \in M_p(N(V, W^*))$ for which $\nu_p(\psi_\alpha) < M$ and the net $\psi_\alpha(v) \in M_p(W^*)$ converges to $\phi(v)$ in norm for all $v \in V$. Then for all $w \in W$,\[
\| S^{(p)}(\psi_\alpha)(v \otimes w) - S^{(p)}(\phi)(v \otimes w) \| = \| \psi_\alpha(v)(w) - \phi(v)(w) \| \\
\leq \| \psi_\alpha(v) - \phi(v) \| \| w \| \rightarrow 0
\]

By linearity, the inequality holds for all $u \in V \hat{\otimes} W$. Since $S$ is a complete isometry, $S^{(p)}(\psi_\alpha)$ and $S^{(p)}(\phi)$ are both bounded. Then we can extend the inequality to whole space $V \hat{\otimes} W$. Thus, $S^{(p)}(\psi_\alpha)(u) \rightarrow S^{(p)}(\phi)(u)$ in norm for all $u \in V \hat{\otimes} W$. Equivalently,

$$(S^{(p)}(\psi_\alpha))^{(n)}(u) \rightarrow (S^{(p)}(\phi))^{(n)}(u)$$

in norm for all $u \in M_q(V \hat{\otimes} W)$ and $q \in \mathbb{N}$. The fact that $\hat{\Phi}$ is the natural complete quotient map $(V^* \hat{\otimes} W^* \rightarrow V^* \hat{\otimes} W^*/\ker \hat{\Phi})$ makes us to be able to choose $F_\alpha \in M_p(V^* \hat{\otimes} W^*)$ such that $\| F_\alpha \| < M$ and $\psi_\alpha = \hat{\Phi}^{(p)}(F_\alpha)$. Considering that the diagram is commutative, we have

$$\| (S^{(p)}(\psi_\alpha))^{(n)}(u) \| = \| (S^{(p)}(\hat{\Phi}^{(p)}(F_\alpha)))^{(n)}(u) \| = \| ((\Phi^*)^{(p)}\theta^{(p)}(F_\alpha))^{(n)}(u) \|.$$ 

Note that the composition $(\Phi^*)^{(p)}\theta^{(p)}(F_\alpha) \in M_p((V \hat{\otimes} W)^*) \cong CB(V \hat{\otimes} W, M_p)$, for any $u \in M_q(V \hat{\otimes} W)$, we have

$$\| ((\Phi^*)^{(p)}\theta^{(p)}(F_\alpha))^{(n)}(u) \| \leq \| ((\Phi^*)^{(p)}\theta^{(p)}(F_\alpha))^{(n)}(u) \|_\nu \leq \| (\Phi^*)^{(p)}\theta^{(p)}(F_\alpha) \|_\nu \leq \| F_\alpha \| \| u \|_\nu \leq M \| u \|_\nu.$$

Thus,

$$\| (S^{(p)}(\psi_\alpha))^{(n)}(u) \| \leq M \| u \|_\nu.$$
By definition of integral map and taking point-norm limit on $\psi_n$, we have

$$\|(S^{(p)}(\phi))^{(q)}(u)\|_{cb} \leq M\|u\|_V$$

for all $q \in \mathbb{N}$. Therefore, $\|(S^{(p)}(\phi)(u))\|_{cb} \leq M\|u\|_V$ for all $u \in V \otimes_{\alpha} W$. Thus,

$$\|S^{(p)}(\phi)\|_{CB(V \hat{\otimes} W, M_p)} \leq M.$$ 

Letting $M = \iota_p(\phi)$, we get

$$\|S^{(p)}(\phi)\|_{CB(V \hat{\otimes} W, M_p)} \leq \iota_p(\phi).$$

Since the diagram is commutative, the uniqueness of $S_I$ is obvious and we have finished. \hfill $\Box$

Given operator spaces $V$ and $W$, we may define a natural complete contraction

$$V \hat{\otimes} W^{**} \to (V \hat{\otimes} W)^{**}$$

by taking the adjoint of the completely isometric injection

$$(V \hat{\otimes} W)^* = CB(V, W^{**}) \hookrightarrow CB(V, W^{**}) = (V \hat{\otimes} W^{**})^*$$

and then taking restriction in $V \hat{\otimes} W^{**}$. We may use (59) to define a linear injection

$$\gamma: V \otimes_{\vee} W^{**} \to V \hat{\otimes} W^{**} \to (V \hat{\otimes} W)^{**} \to (V \otimes_{\vee} W)^{**},$$

where $(V \hat{\otimes} W)^{**} \to (V \otimes_{\vee} W)^{**}$ is induced by taking double dual of the natural complete contraction $V \hat{\otimes} W \to V \otimes_{\vee} W$. Grothendieck proved that the analogous mapping for Banach spaces is always an isometric injection (see [14]). But for operator space case, Archbold and Batty have proved that even if $V$ and $W$ are $C^*$-algebras, $\gamma$ need not be isometric (see [1]). This problem is closely linked to that of determining when $S_I: I(V, W^{**}) \to (V \otimes_{\vee} W)^*$ is a completely isometric bijection. It is described in the following theorem, which is first proved by Z.-J. Ruan in paper [12].

**Theorem 6.3.11.** Let $V$ and $W$ be operator spaces. Then

$$S_I: I(V, W^{**}) \to (V \otimes_{\vee} W)^*$$

is a completely isometric bijection if and only if the mapping

$$\gamma: V \otimes_{\vee} W^{**} \to (V \otimes_{\vee} W)^{**}$$


is a completely isometric injection.

PROOF. We first suppose that $S_I$ is a completely isometric bijection and prove that $\gamma$ is a completely isometric injection.

At first, we have the following commutative diagram

\[ \begin{array}{ccc}
(V \otimes W)^{**} & \xrightarrow{\gamma} & (V^* \otimes W^*)^* \\
\downarrow & & \downarrow \\
V \otimes_v W^{**} & \xrightarrow{i} & (V^* \otimes W^*)^*
\end{array} \]

where the inclusion $i$ is completely isometric and $\delta^*$ is a complete contraction. So, $\gamma$ must increase the matrix norm. In the following, we show that it also decreases the matrix norm.

Note that we have the following commutative diagram of complete contractions

\[ \begin{array}{ccc}
I(V,W^*) & \xrightarrow{S_I} & (V \otimes^v W)^* \\
\downarrow & & \downarrow \\
I(V,W^{**}) & \xrightarrow{\tilde{S}_I} & (V \otimes^v W^{**})^*
\end{array} \]

where we use a tilde to distinguish the corresponding map on a different space. By Proposition 6.3.9, the first column is an isometric injection, it follows that if $S_I$ is a completely isometric bijection, then we obtain a commutative diagram of complete contractions

\[ \begin{array}{ccc}
(V \otimes^v W)^* & \xrightarrow{\varphi^*} & (V \hat{\otimes} W)^* \\
\downarrow & & \downarrow \\
(V \otimes^v W^{**})^* & \xrightarrow{\tilde{\varphi}^*} & (V \hat{\otimes} W^{**})^*
\end{array} \]

where $\eta = S_I^{-1} \circ \tilde{S}_I(S_I$ is a bijection). By taking adjoints we obtain the commutative diagram

\[ \begin{array}{ccc}
V \otimes_v W^{**} & \longrightarrow & (V \hat{\otimes} W^{**})^{**} \\
\downarrow & & \downarrow \\
(V \hat{\otimes} W)^{**} & \longrightarrow & (V \otimes^v W)^{**}
\end{array} \]

where $V \otimes_v W^{**} \longrightarrow (V \hat{\otimes} W^{**})^{**}$ is the natural inclusion map. The bottom composition of maps is just $\gamma$. Since $V \otimes_v W^{**} \longrightarrow (V \otimes^v W^{**})^{**}$ is an operator space
embedding and 

\((V \otimes^\vee W)^{**} \to (V \otimes^\vee W)^{**}\)

is a complete contraction, the composition must be completely contractive. Since the diagram is commutative, we conclude that \(\gamma\) is completely contractive, and so is completely isometric.

Conversely, assume that \(\gamma\) is a completely isometric injection. We also identify 

\((V \otimes^\vee W)^*\) with a vector subspace of \((V \otimes^\vee W)^*\).

Thus, given a function \(F \in M_p((V \otimes^\vee W)^*)\), we have 

\[ F = S^\otimes(\phi) \]

for a unique mapping 

\[ \phi \in M_p(CB(V, W^*)) = CB(V, M_p(W^*)) \]

and it suffices to show that \(\iota_p(\phi) \leq \|F\|_{cb}\).

We may regard \(F\) as a weak* continuous function from \((V \otimes^\vee W)^{**}\) into \(M_p\) (see Theorem 6.1.7). By assumption, \(\gamma\) is a completely isometric injection. So, we may use it to identify \(V \otimes^\vee W^{**}\) with an operator subspace of \((V \otimes^\vee W)^{**}\). Restricting \(F\) to \(V \otimes^\vee W^{**}\) and then extend it to \(V \otimes^\vee W^{**}\), we obtain an extension \(F_1\) of \(F\) with 

\[ \|F_1\| = \|F\|. \]

By the canonical complete isometry \(\tilde{S} : CB(V, W^{**}) \cong (V \otimes^\vee W^{**})^*\), there is a unique mapping 

\[ \phi_1 \in M_p(CB(V, W^{**})) = CB(V, M_p(W^{**})) \]

with \(F_1 = \tilde{S}^\otimes(\phi_1)\). The continuity property for \(F_1\) implies that for each \(v \in V\), 

\[ w^{**} \mapsto \phi_1(v)(w^{**}) = F_1(v \otimes w^{**}) \in M_p \]

is weak* continuous. Identifying \(W\) with an operator subspace of \(W^{**}\), we have that 

\[ \phi_1(v)(w) = F_1(v \otimes w) = F(v \otimes w) = \phi(v)(w) \]

(62)

for all \(w \in W\). We may also identify \(\phi(v) : W \to M_p\) with a weak* continuous map 

\(\phi(v) : W^{**} \to M_p\), and since \(W\) is weak* dense in \(W^{**}\), we conclude from (62) that 

\(\phi_1(v) = \phi(v)\) for all \(v \in V\), that is, \(\phi_1 = \phi\). In the following, we prove that there exists a net \(\psi_\alpha\) converging to \(\phi_1\) with 

\[ \nu_p(\psi_\alpha) < \|F\|_{cb} \]

and so \(\psi_\alpha \to \phi_1 = \phi\) with 

\[ \nu_p(\psi_\alpha) < \|F\|_{cb}, \text{ i.e., } \iota_p(\phi) \leq \|F\|_{cb}. \]

Fixing Hilbert spaces \(H\) and \(K\) with \(V \subseteq V^{**} \subseteq B(H)\) and \(W \subseteq W^{**} \subseteq B(K)\), we have that 

\[ V \otimes^\vee W \subseteq V \otimes^\vee W^{**} \subseteq B(H \otimes K). \]
We may extend $F_1$ to a linear map $F_2 : B(H \otimes K) \to M_p$ with
\[
\|F_2\|_{cb} = \|F_1\|_{cb} = \|F\|_{cb}.
\]
From the identification $(M_p(B(H \otimes K)_*))^{**} \cong M_p((B(H \otimes K)_*)^{**})$, we may approximate
\[
F_2 \in CB(B(H \otimes K), M_p) = M_p(B(H \otimes K)^*) = (M_p(B(H \otimes K)_*))^{**}
\]
by maps $G \in M_p(B(H \otimes K)_*) \subseteq (M_p(B(H \otimes K)_*))^{**}$ with $\|G\|_{cb} < \|F_2\|_{cb}$ in the weak* topology. For each $G$, we let $\hat{G} = G|_{V \otimes W}$, and we then have that $\hat{G} = (S^{-1})^{(p)}(\psi)$ for a unique map
\[
\psi \in M_p(CB(V, W^*)) = CB(V, M_p(W*)�),
\]
(where we once again view $(V \otimes W)^*$ as a vector subspace of $(V \otimes W)^*$). The corresponding net of maps $\psi$ converges to $\phi_1 = \phi$ in the point-weak topology. So, it is also true that $\psi$ converges to $\phi_1 = \phi$ in the point-norm topology (see [12, Lemma 12.3.1]).

On the other hand,
\[
G \in M_p(B(H \otimes K)_*) = CB^*(B(H \otimes K), M_p).
\]
Note that $V \otimes W$ is a subspace of $B(H \otimes K)$. So, $\hat{G}$ is weak* continuous on $V \otimes W$. Meanwhile, from the embedding $V \otimes W \hookrightarrow B(H \otimes K)$, we see that
\[
\hat{G} \in CB(V \otimes W, M_p) = M_p((V \otimes W)^*) \hookrightarrow M_p((V^* \otimes W^*)^{**}).
\]
Therefore, $\hat{G} \in M_p(V^* \otimes W^*)$. Now, $\|\hat{G}\|_{A} \leq \|G\|_{A} \leq \|G\|_{cb}$. Since (58) commutes, we must have that $\psi = \hat{G}^{(p)}(\hat{G})$ (since $\hat{G} = (S^{-1})^{(p)}(\psi)$), and $\nu_p(\psi) < \|F\|_{cb}$. From the latter, we conclude that $\tau_p(\phi) \leq \|F\|_{cb}$.

When $S_I : I(V, W^*) \to (V \otimes W)^*$ is a completely isometric bijection is closely linked to the completely local reflexivity of $W$.

Recall that a Banach space $X$ is called reflexive if the canonical embedding $X \hookrightarrow X^{**}$ is a bijection. $X$ is called locally reflexive if any contraction $\phi : L \to X^{**}$ with $L$ finite dimensional, may be approximated in the point-weak* topology by a net of
finite rank contractions \( \phi_\alpha : L \to X \). Banach spaces are always locally reflexive ([21]).

**Definition 6.3.12.** An operator space \( W \) is said to be completely locally reflexive if any complete contraction \( \phi : L \to W^{**} \) with \( L \) finite dimensional, may be approximated in the point-weak* topology by a net of finite rank complete contractions \( \phi_\alpha : L \to W \).

Not all operator spaces are completely locally reflexive. For example, \( B(l_2) \) is not completely locally reflexive (see [8, page 124-125]). For completely local reflexivity of operator space, we have the following theorem (see [11, Theorem 2.1]).

**Theorem 6.3.13.** If \( V \) is an operator space, then \( V \) is completely locally reflexive if and only if \( L^* \hat{\otimes} V^* \cong (L \hat{\otimes} V)^* \) is a complete isometry for all finite dimensional operator space \( L \).

With the completely local reflexivity, we can give another equivalent condition for which the map \( S_I \) is a completely isometric bijection.

**Theorem 6.3.14.** Given operator space \( W \). Then for any operator space \( V \), the map \( S_I : I(V,W^*) \to (V \hat{\otimes} W)^* \) is a completely isometric bijection if and only if \( W \) is completely locally reflexive.

**Proof.** We first suppose that \( W \) is completely locally reflexive. Then by Theorem 6.3.13, we have the complete isometry \( L^* \hat{\otimes} W^* \cong (L \hat{\otimes} W)^* \) for any finite-dimensional operator space \( L \). By taking dual, we see that \( W \) is completely locally reflexive if and only if \( L \hat{\otimes} W^{**} \cong (L \hat{\otimes} W)^{**} \) is a complete isometry for any finite-dimensional operator space \( L \) (see [10]). By Theorem 6.3.11, we need only to show that \( \gamma : V \hat{\otimes} W^{**} \hookrightarrow (V \hat{\otimes} W)^{**} \) is a completely isometric injection. For any \( u \in M_p(V \hat{\otimes} W^{**}) \), we may regard \( u \) as an element of \( M_p(V_0 \hat{\otimes} V^{**}) \) for some finite-dimensional subspace \( V_0 \subseteq V \). By assumption, we have an isometry \( \gamma_p : M_p(V_0 \hat{\otimes} V^{**}) \cong M_p((V_0 \hat{\otimes} V)^{**}) \). Since the natural embedding

\[
(V_0 \hat{\otimes} V)^{**} \hookrightarrow (V \hat{\otimes} W)^{**}
\]

is completely isometric, we see that \( \|\gamma_p(u)\| = \|u\| \).
6.3. OPERATOR DUAL OF INJECTIVE TENSOR PRODUCT

On the contrary, if \( S_I \) is a completely isometric bijection, then by Theorem 6.3.11, \( V \otimes_v W^{**} \rightarrow (V \hat{\otimes} W)^{**} \) is a completely isometric injection for any operator space \( V \). So, for the particular case that \( V \) is a finite-dimensional operator space \( L \), we get the completely isometric injection \( L \otimes_v W^{**} \rightarrow (L \hat{\otimes} W)^{**} \). Note that \( L \otimes W^{**} \) is norm dense in \( (L \otimes W)^{**} \) for \( L \) being finite-dimensional, we thus have complete isometry \( L \hat{\otimes} W^{**} \cong (L \otimes W)^{**} \).

\[ \square \]

**Remark 6.3.15.** Comparing Theorem 6.3.11 and Theorem 6.3.14 with Theorem 6.2.10, people may wonder why the isometry \( S_I : I(V, W^*) \rightarrow (V \otimes^h W)^* \) is always true for Banach spaces \( V \) and \( W \), but the corresponding complete isometry \( S_I : I(V, W^*) \rightarrow (V \hat{\otimes} W)^* \) is not always true for operator spaces \( V \) and \( W \). We think that the key reason is that, in Banach space case, any Banach space is automatically locally reflexive, but not all complete operator spaces are completely locally reflexive.

**Examples** In order to see under what conditions, the map \( S_I : I(V, W^*) \rightarrow (V \hat{\otimes} W)^* \) is a completely isometric bijection, it suffices to check if \( W \) is a completely locally reflexive by Theorem 6.3.14.

1. it is easy to see that any (completely) reflexive operator space is completely locally reflexive (e.g., any finite-dimensional operator space \( L \));

2. Let \( A \) and \( B \) be \( C^* \)-algebras, and let \( A \otimes B \) be their algebraic tensor product. In general, there is no way of giving the involutive algebra \( A \otimes B \) a norm such that it is a \( C^* \)-algebra, but it is always possible to give it a norm such that it has all the properties of a \( C^* \)-algebra except for completeness (see [17]). Such a norm is called a \( C^* \)-norm on \( A \otimes B \). A \( C^* \)-algebra \( A \) is said to be nuclear if for any \( C^* \)-algebra \( B \), there is a unique \( C^* \)-norm on \( A \otimes B \). Then, all nuclear \( C^* \)-algebras are completely locally reflexive (e.g., \( K(H) \) or commutative \( C^* \)-algebras) (see [10]).

3. all preduals of von Neumann algebras, especially duals of \( C^* \)-algebras are completely locally reflexive (e.g., \( T(H) = K(H)^* = (B(H))^* \) (see [8, Proposition 5.4]).

As a particular example of the third case, let \( I \) be any index set and \( T_I = T(l_2(I)) \) be the space of all trace class operators on \( l_2(I) \). Then \( T_I \) is an operator space with the canonical operator space matrix norm obtained by identifying \( T_I \) with the operator predual of the operator space \( B(l_2(I)) \) of all bounded linear operators. Then
$T_I$ is completely locally reflexive (see [10] and [16]). Thus, for any operator space $V$, $S_I : I(V, T_I^*) \rightarrow (V \otimes^\gamma T_I)^*$ is a complete isometry.
Bibliography


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